Discrete Structures

Fight for Survival

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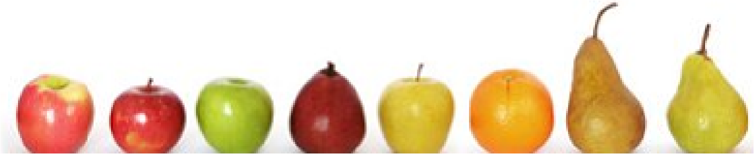
PERMUTATION & COMBINATION

Permutation

Try to solve this one.

How many ways are there to arrange these fruits?



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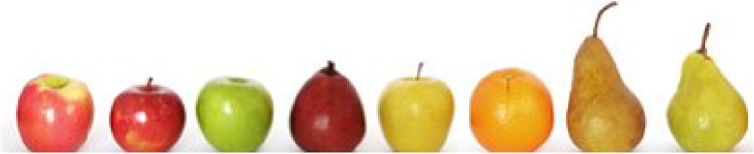

Remember product rule.

$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8!$

Permutation

In how many ways, we can make 3 people stand for a photo out of these people?

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$29 \times 28 \times 27$

Permutation - Understanding

Let's understand the concept of permutation.

Permutation:

Suppose, we have a list of n objects (an ordered set, where it is specified which element is the first, second, etc.), and we rearrange them so that they are in another order, this is called **permuting** them; the new order is called a **permutation** of the objects. An ordered arrangement of r elements of a set is called an **r-permutation**

Example:

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Of all permutations for some three distinct intgers, e.g., $\{1, 2, 3\}$, what fraction have first integer smaller than the last?

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Example:

Of all permutations for some three distinct intgers, e.g., $\{1, 2, 3\}$, what fraction have first integer smaller than the last?

Exactly half!

Permutation - Understanding

So, the question is to determine the number of ways n objects can be ordered. The solution can be found in general:

- We can put any of the n people in the first place; no matter whom we choose, we have $n-1$ choices for the second.
- So, the number of ways to fill the first two positions is $n.(n-1)$.
- No matter how we have filled the first and second positions, there are $n-2$ choices for the third position, so the number of ways to fill the first three positions is $n.(n-1).(n-2)$.
- It is clear that this argument goes on like this until all positions are filled.
- Thus the number of ways to fill all positions is $n.(n-1).(n-2)....3.2.1 = n!$ the factorial of n.

Permutation - Theorem

The number of permutations of n objects is $n!$

Permutation - Ordered Subsets

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So, the first two prizes can go 100×99 ways.
Given the first two, there are 98 athletes who can be third etc.

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So, the first two prizes can go 100×99 ways.
Given the first two, there are 98 athletes who can be third etc.

Therefore, by product rule:
$100 \times 99 \times ... \times 91$
$100 \times 99 \times 98 \times \ldots \times (100 - 10 + 1)$

Permutation - Ordered Subsets - Theorem

Let's generalize it for having k elements subset from an n element set.

The number of ordered $k-element$ subsets of a set with n elements is $n \times (n-1) \times (n-2) \times ... \times (n-k+1)$

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We may also write it like:

$= n \times (n-1) \times (n-2) \times \ldots \times (n-k+1) \times \frac{(n-k)!}{(n-k)!}$

$= \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-k+1) \times (n-k)(n-k-1) \times \ldots 3 \times 2 \times 1}{(n-k)!}$

$= \frac{n!}{(n-k)!}$

Permutation

So,

Permutation:

If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n,r) = n \times (n-1) \times (n-2) \times ... \times (n-r+1)$$

$r - permutations$ of a set with n distinct elements.

If n and r are integers with $0 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Permutation - Example

Question: In how many ways can we select three students from a group of five students to stand in a line for a photo?

Question: In how many ways can we arrange all five students for a photo?

Permutation - Example

Question: In how many ways can we select three students from a group of five students to stand in a line for a photo?

Answer: $P(5,3) = \frac{5!}{(5-3)!} = \frac{5\times4\times3\times2!}{2!} = 5 \times 4 \times 3 = 60$

Question: In how many ways can we arrange all five students for a photo?

$P(5,5) = \frac{5!}{(5-5)!} = \frac{5\times4\times3\times2\times1}{0!} = 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$

Permutation - Example

Question: How many permutations of the letters ABCDEFGH exist?

Question: How many permutations of the letters ABCDEFGH contains the string ABC?

Permutation - Example

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Answer:
$P(8,8) = \frac{8!}{(8-8)!} = \frac{8\times7\times6\times5\times4\times3\times2\times1}{0!} = 8\times7\times6\times5\times4\times3\times2\times1 = 8! = 40,320$

Question: How many permutations of the letters ABCDEFGH contains the string ABC?

Consider the string ABC as a single unit and we have
$P(6,6) = \frac{6!}{(6-6)!} = \frac{6\times5\times4\times3\times2\times1}{0!} = 6\times5\times4\times3\times2\times1 = 6! = 720$

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Answer: The answer is the number of 3-permutations of a set of 100 elements. Consequently, the answer is

$$P(100,3) = \frac{100!}{(100-3)!} = \frac{100\times 99\times 98\times 97!}{97!} = 100\times 99\times 98 = 970,200$$

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Now imagine we want to give three prizes but we do not want to distinguish between first, second, and third prize. How many ways are there?

Permutation - Example

Now imagine we want to give three prizes but we do not want to distinguish between first, second, and third prize. How many ways are there?

Answer: The answer is the number of 3-permutations of a set of 100 elements modulo the order of the three people we select.

Consequently, the answer is

$$P(100,3)/3! = \frac{100!}{(100-3)!3!} = \frac{100\times 99\times 98\times 97!}{97!}/3! = 970,200/6$$

Combination - Subsets (un-ordered)

We can easily drive, from here, one of the most useful counting result.

The number of ordered $k-element$ subsets is given by:
$n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)$

The number of $k-element$ subsets (un-ordered) of an $n-element$ set is

$$\binom{n}{k} = \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)}{k!}$$

$$= \frac{n!}{k!(n-k)!}$$

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We have a special notation for this "n choose k":

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Combination

So,

Combination:

The number of $r - combination$ of a set with n elements, where n is a non-negative integer and r is an integer with $0 \leq r \leq n$, equals

$$\binom{n}{k} = \frac{n!}{r!(n-r)!}$$

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The number of $r-combination$ of a set with n elements, where n is a non-negative integer and r is an integer with $0 \leq r \leq n$, equals

$$\binom{n}{k} = \frac{n!}{r!(n-r)!}$$

$$P(n,r) = \binom{n}{k}.P(r,r)$$

$$\binom{n}{k} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

Combination - Example

Question: How many different committees of three students can be formed from a group of four students?

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Question: How many different committees of three students can be formed from a group of four students?

Answer: Because, in a committee, the order of the members does not matter, Consequently, the answer is

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4\times 3!}{3!\times 1!} = 4$$

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Answer: The answer is given by the number of 5-combinations of a set with 10 elements. So, the number of such combinations is

$$\binom{10}{5} = \frac{10!}{5!(10-5)!} = \frac{10\times9\times8\times7\times6\times5!}{5!\times5!} = 252$$

Combination - Example

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Question: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Answer:The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. So, the number of such combinations is

$$\binom{30}{6} = \frac{30!}{6!(30-6)!} = 593,775$$