Discrete Structures

Fight for Survival

Dr. Mudassir Shabbir

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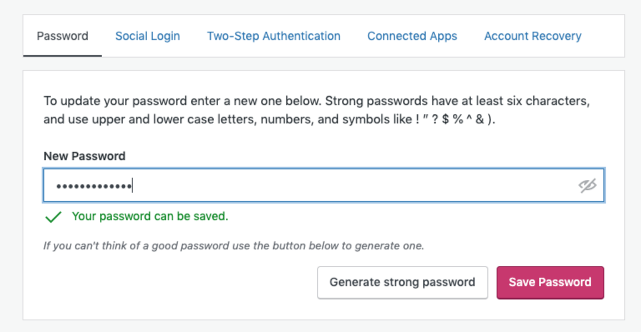
COUNTING / COMBINATORICS

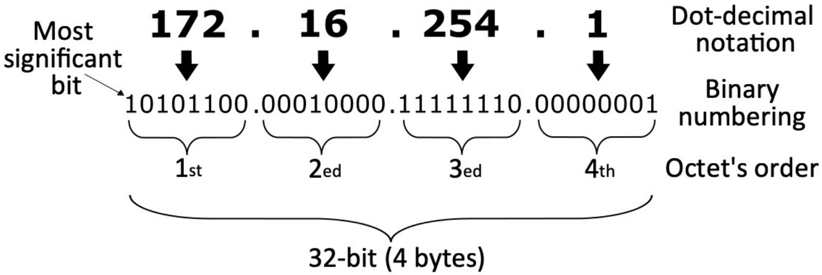
Counting

You want to secure your Instagram. Hackers have a program that can try one million passwords per second. How long would it take them to break your password with the following rules?



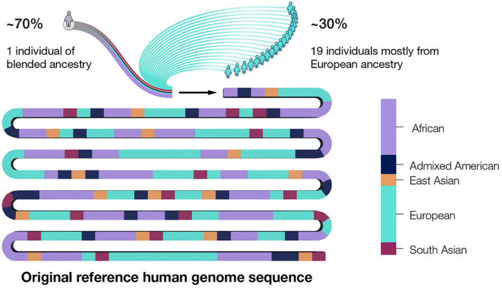
Counting

How many unique internet devices are possible?



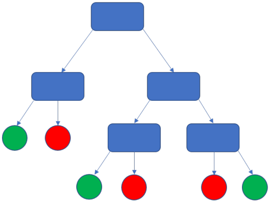
Counting

Given a collection of DNA sequences which two are most likely to respond to the same medicine?

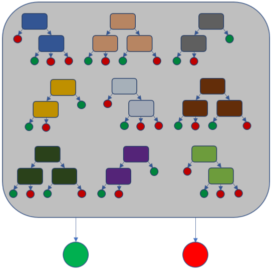


Counting in ML

What is the optimal height of a Decision Tree?



Decision Tree



Random Forest

Counting

So, let us start Counting!

- We will start with some basic rules.
- Followed by Some Practice Examples.
- We see how to count Combinations and Permutations.
- And Lastly, we will see Pigeonhole Principle.

The Product Rule

Basic counting questions are of the form

In how many ways can you do this task?

The "product rule" applies when a task is made up of separate tasks that can be done in series.

If a task can be broken down into two tasks with n ways to do the first task and m ways to do the second, then there are $n \times m$ ways in total.

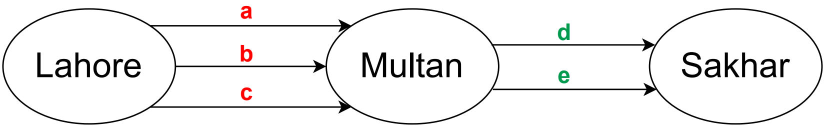
Theorem

$$|A \times B| = |A| \times |B|$$

The Product Rule

If a task can be broken down into two tasks with n ways to do the first task and m ways to do the second, then there are $n \times m$ ways in total.

The Product Rule - Understanding



Question: How many ways to go from Lahore to Sakhar?

One can go from Lahore to Sakhar by using the following paths

$a - d$, $a - e$, $b - d$, $b - e$, $c - d$ or $c - e$

OR

Total ways = ways from Lahore to Multan $\times$ways from Multan to Sakhar

Total ways = 3 $\times$2 = 6

Generalized Product Rule

Let's generalize the product rule for finite number of tasks.

Generalized Product Rule:

Suppose that a procedure can be broken down into a sequence of i tasks. If there are n_1 ways to do the first task, for each of these ways of doing the first task, there are n_2 ways to do the second task, and for all of these ways of doing the first $i-1$ tasks, there are n_i ways to do the i^{th} task, then there are $n_1 \times n_2 \times ... \times n_i$ ways to do the procedure.

The Product Rule - Example

Question: How many different bit strings of length eight are there?

0 1 1 1 0 1 1 0

0 1 0 1 0 1 1 0

0 1 1 1 1 1 1 0

The Product Rule - Example

Question: How many different bit strings of length eight are there?

Answer: Each of the eight bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^8 = 256$ different bit strings of length eight.

0 1 1 1 0 1 1 0

0 1 0 1 0 1 1 0

0 1 1 1 1 1 1 0

The Product Rule - Example

Question: A new company with just two employees, Fatima and Ali, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

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Answer: The procedure of assigning offices to these two employees consists of assigning an office to Fatima, which can be done in 12 ways, then assigning an office to Ali different from the office assigned to Fatima, which can be done in 11 ways. By the product rule, there are $12 \times 11 = 132$ ways to assign offices to these two employees.

The Product Rule - Example

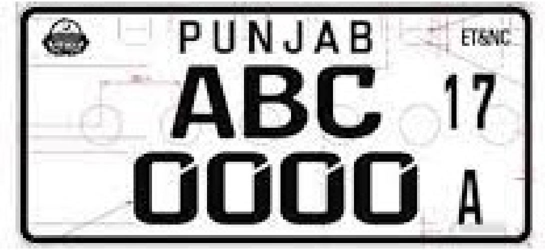
Question: The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

The Product Rule - Example

Question: The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

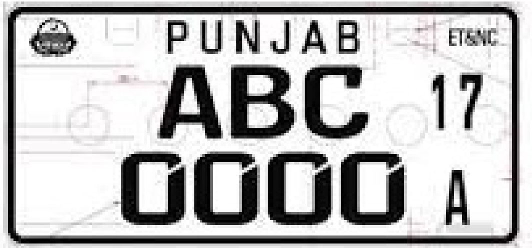
Answer: The product rule shows that there are $26 \times 100 = 2600$ different ways that a chair can be labeled.

The Product Rule - Example



Question: In Pakistan, the registration of a car tags the car with 3 capital English letters followed by 4 digits. How many possible car tags are there to register a car?

The Product Rule - Example

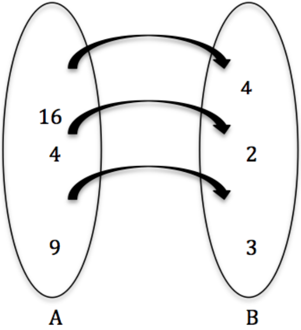


Question: In Pakistan, the registration of a car tags the car with 3 capital English letters followed by 4 digits. How many possible car tags are there to register a car?

Answer: There are 26 choices for each of the three capital English letters and ten choices for each of the three digits. Hence, by the product rule there are a total of
$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$ possible car tags.

The Product Rule - Example

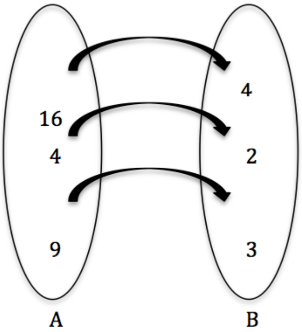
Question: How many $one-to-one$ functions are there from a set with m elements to one with n elements?



The Product Rule - Example

Question: How many $one - to - one$ functions are there from a set with m elements to one with n elements?

Answer: First note that when $m > n$ there are no $one - to - one$ functions from a set with m elements to a set with n elements.



The Product Rule - Example

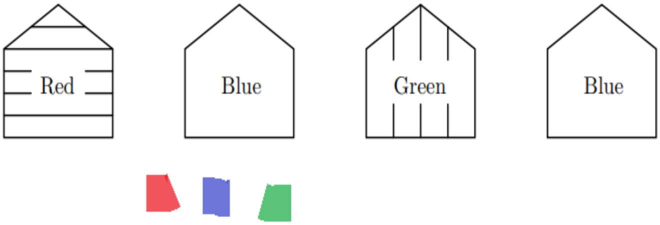
Question: How many $one-to-one$ functions are there from a set with m elements to one with n elements?

Answer:
Now let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, ..., a_m$.
There are n ways to choose the value of the function at a_1.
Because the function is $one-to-one$, the value of the function at a_2 can be picked in $n-1$ ways (because the value used for a_1 cannot be used again). In general, the value of the function at a_k can be chosen in $n-k+1$ ways. By the product rule, there are $n(n-1)(n-2)\cdots(n-m+1)$ $one-to-one$ functions from a set with m elements to one with n elements.
For example, there are $5 \times 4 \times 3 = 60$ $one-to-one$ functions from a set with three elements to a set with five elements.

Product Rule

Try to solve this one.

In how many ways you can color 4 houses using 3 colors - red, blue, green?



Product Rule

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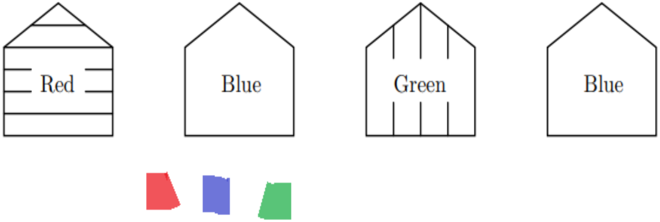

Remember product rule.

$3 \times 3 \times 3 \times 3 = 3^4$

The Sum Rule

The sum rule is applied when there are multiple choices but only one selection is made.

If a task can be done either in one of n ways OR in one of m ways, where none of the set of n ways is the same as any of the set of m ways, then there are $n+m$ ways to do the task.

Sum Rule says,

$$|A \cup B| = |A| + |B| \text{ whenever } A \cap B = \emptyset$$

The Sum Rule - Example

Question: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

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Answer: The student can choose a project by selecting a project from the first list, the second list, or the third list. Because no project is on more than one list, by the sum rule there are $23 + 15 + 19 = 57$ ways to choose a project.

The Sum Rule - Example

Question: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

The Sum Rule - Example

Question: Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Answer: There are 37 ways to choose a member of the mathematics faculty and there are 83 ways to choose a student who is a mathematics major. Choosing a member of the mathematics faculty is never the same as choosing a student who is a mathematics major because no one both a faculty member and a student. By the sum rule it follows that there are $37 + 83 = 120$ possible ways to pick this representative.

An Interesting Counting Problem

I AM PLANNING ASSESSMENT METHODS FOR THE NEXT SEMESTER AND I HAVE AN INTERESTING COUNTING PROBLEM FOR YOU!

An Interesting Counting Problem

I like quizzes more than homeworks but there is a trade off. Quiz is easy to manage and it is fair. Homeworks, on the other hand, provide better opportunity to learn through discussion.

An Interesting Counting Problem

Enter, quizzes in pairs! Avoids plagiarism and fosters discussion.

An Interesting Counting Problem

Enter, quizzes in pairs! Avoids plagiarism and fosters discussion.

However, I still observe some piggy-backing.

An Interesting Counting Problem

So, naturally, I add the condition that you can not partner with a person twice, i.e., in every quiz you have to pick a new partner.

Problem

In a class of n (assume n is even) students, if every student is asked to pair up with new partner for every quiz, how many quizzes are possible?

Principle of Inclusion-Exclusion (PIE)

Principle of inclusion-exclusion, also known as the subtraction rule, is defined as

Subtraction Rule:

If a task can be done either in n_1 ways or in n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Recall this from sets!

PIE - Example

Question: How many bit strings of length 8 are there that either starts with 1 or ends with 00?

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Answer:

- Bit stings starts with 1: $1*******=2^7=128$
- Bit stings ends with 00: $******00=2^6=64$
- Bit stings starts with 1 or ends with $00=128+64=192$???

PIE - Example

Question: How many bit strings of length 8 are there that either starts with 1 or ends with 00?

Answer:

- Bit stings starts with 1: $1 * * * * * * * = 2^7 = 128$
- Bit stings ends with 00: $* * * * * * 00 = 2^6 = 64$
- Bit stings starts with 1 or ends with $00 = 128 + 64 = 192$???

We are over-counting

Bit stings starts with 1 and ends with 00: $1 * * * * * 00 = 2^5 = 32$

counted twice

Bit stings starts with 1 or ends with $00 = 128 + 64 - 32 = 160$

PIE - Example

There are 40 students in the class.

- Babar Azam fans - 18
- Shaheen Afridi fans - 16
- Muhammad Rizwan fans - 12
- Babar Shaheen fans - 7
- Babar Rizwan fans - 5
- Shaheen Rizwan fans - 3
- B S R fans - 2

PIE - Example

Questions:

1. How many students are fans of any cricketer?
2. How many students are not fans of any cricketer?

PIE - Example

Questions:

1. How many students are fans of any cricketer?
   $18 + 16 + 12 - 7 - 5 - 3 + 2 = 33$
2. How many students are not fans of any cricketer?
   $40 - 33 = 7$

PIE - Example

Each user on a computer system has password, which is six characters long, where each character is an upper-case letter or a digit. Each password must contain at least one digit. How many passwords are there?

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Answer:
Total possible passwords : $36 \times 36 \times 36 \times 36 \times 36 \times 36$

PIE - Example

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Answer:
Total possible passwords : $36 \times 36 \times 36 \times 36 \times 36 \times 36 = 36^6$
Illegal passwords: Only have characters... Not digits :
$26 \times 26 \times 26 \times 26 \times 26 \times 26 = 26^6$
Total legal passwords : Total passwords - Illegal passwords
$36^6 - 26^6$

PIE - Example

How many positive integers, not bigger than 20, are divisible by 2 or 3?

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Answer:
Numbers divisible by $2 = 20/2 = 10$
Numbers divisible by $3 = \lfloor 20/3 \rfloor = 6$
Numbers divisible by both 2 and $3 = \lfloor 20/(2 \times 3) \rfloor = 3$

Numbers divisible by 2 or $3 = 10 + 6 - 3 = 13$

PIE - Class Activity

How many positive integers up to 1000 are divisible by 7 or 11?

How many positive integers up to 1000 are not divisible by 7 or 11?

PIE - Class Activity

How many positive integers up to 1000 are divisible by 7 or 11?

How many positive integers up to 1000 are not divisible by 7 or 11?

Answer:
Numbers divisible by $7 = \lfloor 1000/7 \rfloor = 142$
Numbers divisible by $3 = \lfloor 1000/11 \rfloor = 90$
Numbers divisible by both 7 and $11 = \lfloor 1000/(7 \times 11) \rfloor = 12$

Numbers divisible by 7 or $11 = 142 + 90 - 12 = 220$
Numbers not divisible by 7 or $11 = 1000 - 220 = 780$

PIE - Class Activity

In a survey on the college students, the following data was obtained:

- 78 like Vanilla Ice-cream
- 32 like Chocolate Ice-cream
- 57 like Mango Ice-cream
- 13 like both Vanilla Ice-cream and Chocolate
- 21 like both Chocolate and Mango Ice-cream
- 16 like both Mango Ice-cream and Vanilla Ice-cream
- 5 like all three flavors above
- 14 like none of these three flavors

How many students were surveyed?

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How many students were surveyed?

Answer:
$78 + 32 + 57 - 13 - 21 - 16 + 5 + 14 =$ Total Students

PIE - Example

Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter, lower case letter or a digit. Each password must contain at least one DIGIT, one UPPER CASE and one LOWER CASE letter.
How many possible passwords are there?

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How many possible passwords are there?

Answer:
Total Passwords = Passwords of length 6 + Passwords of length 7 + Passwords of length 8

If we can find passwords of length 6, we can similarly find passwords of 7 and 8.

Let's focus on passwords of length 6.

PIE - Example

Each user on a computer system has a password, which is **six** characters long, where each character is an uppercase letter, lower case letter or a digit. Each password must contain at least one DIGIT, one UPPER CASE and one LOWER CASE letter.
How many possible passwords are there?

Answer:
Total Passwords = Possible Passwords - Illegal Passwords

Possible Passwords: 62^6
Illegal Passwords: (Only upper or lower case letters) + (Only upper case or digits) + (Only lower case or digits) - (Only upper case letters) - (Only lower case letters) - (Only digits) =
$52^6 + 36^6 + 36^6 - 26^6 - 26^6 - 10^6$
Total Passwords = $62^6 - 52^6 - 36^6 - 36^6 + 26^6 + 26^6 + 10^6$