Discrete Structures

Fight for Survival

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Functions

Functions

Some terminology:

For a binary relation R from set A to B, let's call set A domain, and set B co-domain. Let's also use the term Range for the subset of B that some element in A relates to.

Consider the relation R defined on set $A = \{1, 2, 3\}$ and set $B = \{a, b, c\}$ as:
$R = \{(1, a), (2, a), (3, b)\}$.

- Domain (Dom(R)): $\{1, 2, 3\}$
- Co-domain (CoDom(R)): $\{a, b, c\}$
- Range (Rng(R)): $\{a, b\}$

Functions

A binary relation is called function if every element in the domain is related to exactly one element of the range, i.e., a R from A to B is a function if $\forall a \in A(\exists b aRb \wedge \forall b, c(aRb \wedge aRc \implies b = c))$

For $A = \{1, 2, 3, 4\}$ and $Z = \{A, B, C\}$, the relation $R = \{(2, A), (3, B), (4, C)\}$ from A to Z is a function.

The function is also defined as:

Let A and B be nonempty sets. A function f from A to B is an assignment or mapping of each element of A to a exactly one element of B. We write aRb as $f(a) = b$. If f is a function from A to B, we write $f : A \to B$

Functions - Some Terminology

if $f : A \rightarrow B$, i.e. $f(a) = b$ (where $a \in A \& b \in B$), then:

- A is the domain of f
- B is the co-domain of f
- b is the image of a under f
- a is the pre-image of b under f
  - Note: b may have more than one pre-images.
- The range $R \subseteq B$

Range vs. Co-domain - Example

Suppose that: f is a function of mapping students in this class to the set of grades $\{A, B, C, D, F\}$

f's co-domain is $\underline{\{A, B, C, D, F\}}$, and the range is $\underline{Un-known}$

Suppose that the grades turn out to be all As and Bs then,

The f's range is $\underline{\{A, B\}}$ but the co-domain is still $\underline{\{A, B, C, D, F\}}$

Functions- One-to-One (Injective)

The one-to-one function could be defined as:

A function $f : A \to B$ is one-to-one (1-1), or injective, or an injection, iff every element of its range has only one pre-image

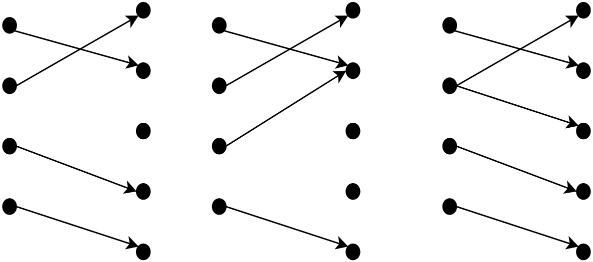
Only one element of the domain is mapped to any given one element of the range.

Question: Domain and co-domain have the same cardinality. What about range?

Question: Domain and range have the same cardinality. What about co-domain?

One-to-One Illustration

Graphical representations of functions that are (or not) one-to-one:



One-to-One Not One-to-One Not a function

Functions- Onto (Surjective)

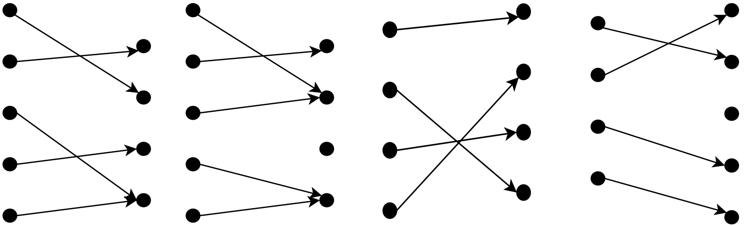
The onto function could be defined as:

A function $f : A \rightarrow B$ is onto, or surjective, or an surjection, iff its range is equal to its co-domain

An onto function maps the set A onto (over, covering) the entirety of the set B, not just over a piece of it.

Onto Illustration

Graphical representations of functions that are (or not) onto:



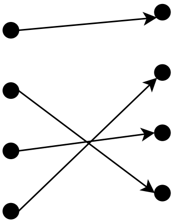
Onto (Not 1-1)	Not Onto (or 1-1)	Both (Onto and 1-1)	1-1 but not Onto

Functions - Bijective

The bijective function could be defined as:

A function $f : A \to B$ is bijective or bijection or invertible or reversible iff it is one-to-one and onto at the same time

we have already seen the following example before:



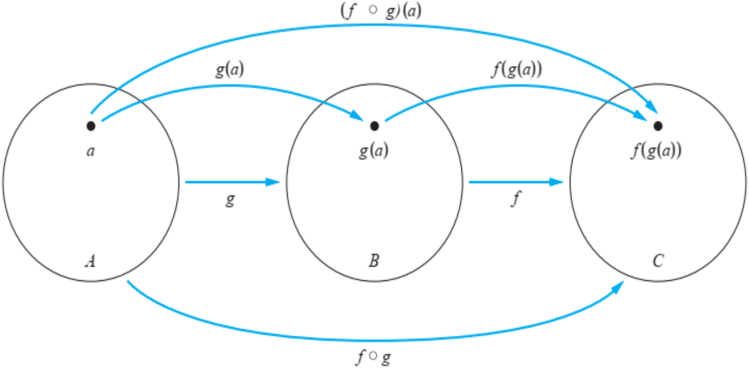
Both 1-1 and onto

Composition of the function

Functions of functions

The composition of the function could be defined as:

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by fog, is defined by
$(fog)(a) = f(g(a))$



Floor and Ceiling Functions

Floor Function:

$$\text{floor}(x) = \lfloor x \rfloor \quad \text{is the largest integer} \leq x$$

Ceiling Function:

$$\text{ceiling}(x) = \lceil x \rceil \quad \text{is the smallest integer} \geq x$$

- $\lfloor 3.7 \rfloor = 3, \quad \lceil 3.7 \rceil = 4$
- $\lfloor -2.3 \rfloor = -3, \quad \lceil -2.3 \rceil = -2$

Question

Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by $f(x) = x^2$. Is f injective?

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No. For example, $f(2) = f(-2)$. So the mapping $f : \mathbb{Z} \to \mathbb{N}$ is not unique for f. Therefore, f is not injective.

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Yes. Since there are more people than seats ($45000 > 35000$), we know that all seats will be filled. Therefore f is surjective.

Question

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No. f is not surjective. There is no mapping for element 3 in $\mathbb{N}$ from f since 3 is not a square.

How to Show a Function is Injective?

To prove $f : A \to B$ is injective

Key Point:

- Let $x \in A$ and $f(x)$ is an image of x.
- Let $y \in A$ and $f(y)$ is an image of y.

For injectivity of f, all we need to show is: If $f(x) = f(y)$, then $x = y$.

How to Show a Function is Injective?

Let X be the set of non-zero real numbers.
Show that: $f : X \to X$ is injective, where $f(x) = \frac{1}{x}$.

Let $x, y \in X$.
Then, $f(x) = \frac{1}{x}$ and $f(y) = \frac{1}{y}$.
(We need to show that $f(x) = f(y)$ implies $x = y$.)

$f(x) = f(y)$ implies that $\frac{1}{x} = \frac{1}{y}$, which further implies that $x = y$.

Hence, f is injective. Q.E.D.

Counting Functions

What is the total number of functions from a set $A, |A| = a$, to a set $B, |B| = b$?

Counting Functions

What is the total number of functions from a set $A, |A| = a$, to a set $B, |B| = b$?

The answer is b^a. Think of an example, like, $A = \{1, 2\}$, then 1 can be mapped to b things in B and so can 2. We have b independent choices for both so it is $b \times b = b^2$.

Counting Functions

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The answer is $\frac{b!}{(b-a)!}$. This time first member of A has b choices but the second member has one less choice, and third member has two less and so on.

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That is a bit tricky, and we need a few new ingredients/tools.

Tool 1: $|A| = |U| - |A`|$

Tool 2: PIE or Principle of Inclusion-Exclusion

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Inclusion-Exclusion for sets:

The number of elements in the union of two sets is the sum of the number of elements in these sets minus the number of elements in their intersection.

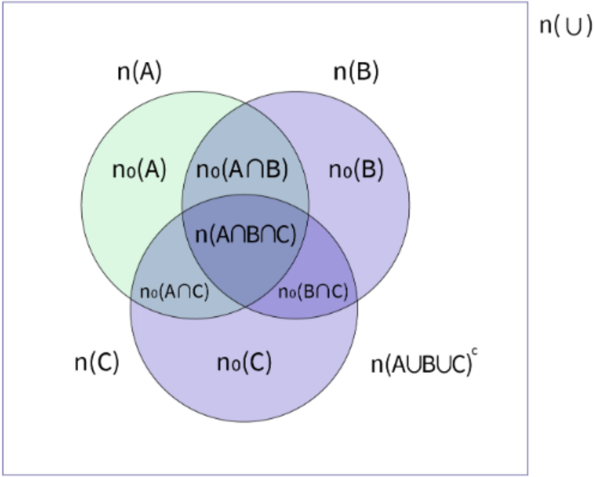
$$|S \cup T| = |S| + |T| - |S \cap T|$$

Tool 2: PIE or Principle of Inclusion-Exclusion

Consider three sets S_1, S_2, S_3 , for counting the total elements in the union of three sets , we will need to know the number of items in set S_1, the number of items in set S_2, the number of items in set S_3 and then we need to know the number of items common in all the three sets S_1, S_2 and S_3. (overlapping of all the three sets).

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_2 \cap S_3| - |S_3 \cap S_1| + |S_1 \cap S_2 \cap S_3|$$

Tool 2: PIE or Principle of Inclusion-Exclusion



Tool 2: PIE or Principle of Inclusion-Exclusion

Consider a group of students out of which 15 study Potions, 11 study DADA and 13 study Charms. The number of students who study all three are 4. The number of students who study both Potions and DADA are 6, The number of students who study both Charms and Potions are 8 and The number of students who study both Charms and DADA are 7. What is the size of the group?

Considering the above scenario we need to find out how many students study either (Potions or DADA or Charms) or all.

Tool 2: PIE or Principle of Inclusion-Exclusion

Lets us use P for Potions and C for DADA and E for Charms, we get the following formula:

$$|P \cup C \cup E| = |P| + |C| + |E| - |P \cap C| - |C \cap E| - |E \cap P| + |P \cap C \cap E|$$
$$|P \cup C \cup E| = 15 + 11 + 13 - 6 - 7 - 8 + 4$$
$$|P \cup C \cup E| = 22$$

Tool 3: Combination

So,

Combination:

The number of $r - combination$ of a set with n elements, where n is a non-negative integer and r is an integer with $0 \leq r \leq n$, equals

$$\binom{n}{k} = \frac{n!}{r!(n-r)!}$$

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$$P(n,r) = \binom{n}{k}.P(r,r)$$

$$\binom{n}{k} = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

Counting Functions

What is the total number of onto or surjective functions from a set $A, |A| = a$, to a set $B, |B| = b$?