Discrete Structures

Fight for Survival

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Mathematical Induction

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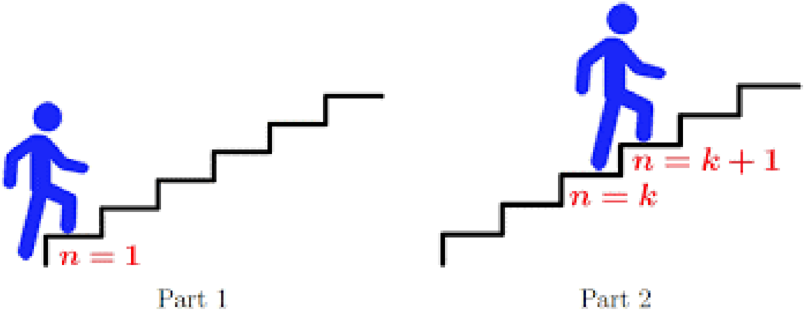
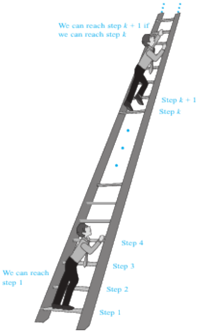


Climbing an Infinite Ladder

Suppose that we have an infinite ladder, we want to know whether we can reach every step on this ladder. We know two things :

a) We can reach the first step of the ladder.
b) If we can reach a particular step of the ladder, then we can reach the next step.



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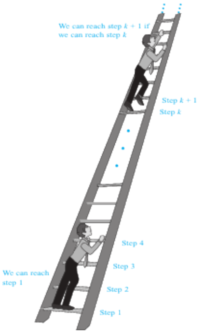

Induction says that these two pieces of information enough to reach the required conclusion!

Climbing an Infinite Ladder

Here is how it works (informally!)

From (a) we can reach the first step. Then by applying (b) we can reach the second step. Applying (b) again, the third step. And so on. We can apply (b) any number of times to reach any particular step, no matter how high up.

After 100 uses of (b), we know that we can reach the 101st step.

Principle of Mathematical Induction

To prove that P(n) is true for all positive integers n, we complete these steps:

Basis Step: Show that P(1) is true.

Inductive hypotheis: We asume that statement is true for some positive integer k

Inductive Step(proof): Based on the assumption, we prove that the statement must be true for $k+1$

To complete the inductive step, assuming the inductive hypothesis that P(k) holds for an arbitrary integer k, show that P(k + 1) must be true.

Proof by Induction: Example 0

Example: Prove the following using induction.

"The sum 1 + 2 + ... + n is equivalent to $\frac{n(n+1)}{2}$."

We need to prove $\forall n \geq 1, P(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

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$P(k) = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Prove by Induction: Example 0

Example: Prove the following using induction.
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Inductive Step(Cont.):

Now, using this assumption, i.e., $P(k) = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$, we need to prove that the statement is also true for $k+1$.

Show that : $P(k+1) = \frac{(k+1)(k+2)}{2}$

$$P(k+1) = 1 + 2 \ldots + k - 1 + k + (k+1) = P(k) + (k+1) = \frac{(k)(k+1)}{2} + (k+1)$$

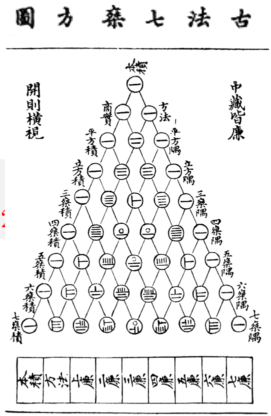
$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$ Q.E.D.

Proof by Induction: Example 1

Example: Prove the following using induction:

"The sum of entries at n^{th} row is



Proof by Induction: Example

Example 2: Prove that for $n \geq 4$, $2^n \geq 3n$.

Proof by Induction: Example

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Base Case:
$n = 4$ (We see if the theorem statement is true or not for the base case)

$2^n = 2^4 = 16 \geq 12 = 3(4)$
Thus, for $n = 4$,

$$2^n \geq 3n \text{ is true.}$$

Inductive Step:
We assume that the theorem is true some $k > 4$, that is,

$$2^k \geq 3k$$

Inductive Step(Cont.):

Next we need to show that the theorem is true for $k+1$. More precisely, we need to show that

If $2^k \geq 3k$, then $2^{k+1} \geq 3(k+1)$

So, lets try to prove the above,

$$\begin{aligned} 2^{(k+1)} &= 2 \times 2^k \\ &\geq 2 \times 3k = 3k + 3k \\ &\geq 3k + 3 \qquad (\text{ because } k \geq 4) \\ &= 3(k+1). \end{aligned}$$

Hence, we showed that $2^{(k+1)} \geq 3(k+1)$, which proves the desired result.

Proof by Induction: Example

Example 3: Prove that for any positive integer n

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Proof by Induction: Example

Example 3: Prove that for any positive integer n
$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$$

Base Case:

$n = 1$ (We see if the theorem statement is true or not for the base case)

L.H.S = $\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{2}$

R.H.S= $\frac{n}{n+1} = \frac{1}{2}$

Hence, the statement is true for $n = 1$.

Proof by Induction..(Cont.)

Example 3: Prove that for any positive integer n
$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

Inductive Hypothesis:

We assume that the theorem is true for $n = k$, that is,

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{(k+1)}$$

Inductive Proof:

Next we need to show that the theorem is true for $n = k + 1$.
More precisely, we need to show that:

If $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{(k+1)}$ is true, then $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{(k+1)}{(k+2)}$

Proof by Induction..(Cont.)

Inductive Proof (Cont.):

If $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{(k+1)}$ is true, then $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{(k+1)}{(k+2)}$

$$
\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{(k+1)}{(k+2)}
\end{aligned}
$$

Strong Induction-Example

Prove "$P(n) = n$ can be written as the product of prime numbers, where n is an integer ≥ 2."

Product of k numbers for $k \geq 2$ is well-defined; for $k = 1$, it is defined as just that one number.

Weak Induction

Prove $P(n)$ for $n \geq 1$

Base Case: Show $P(1)$ is true

Inductive Hypothesis:

a) Pick some k

b) Assume $P(i)$ holds for $i = k$, i.e., $P(k)$ is true for $i = k$.

Inductive Step: Show that $P(k) \to P(k+1)$

Strong Induction

Prove $P(n)$ for $n \geq 1$

Base Case: Show $P(1)$ is true

Inductive Hypothesis:

a) Pick some k

b) Assume $P(i)$ holds for all $i \leq k$. i.e., $P(i)$ is true $\forall i \leq k$

Inductive Step: Show that $P(k) \rightarrow P(k+1)$

Strong and Weak Induction

Difference between Strong and Weak Induction

The only difference is in inductive hypothesis step.

In weak induction, we only assume that particular statement holds at k^{th} step, while in strong induction, we assume that the particular statement holds at all the steps from the base case to k^{th} step.

Strong Induction-Example

Prove "$P(n) = n$ can be written as the product of prime numbers, where n is an integer ≥ 2."

Strong Induction-Example

Prove "$P(n) = n$ can be written as the product of prime numbers, where n is an integer ≥ 2."

Base Case: $n = 2$
2 is a prime, so it is a product of a single prime. P(2) holds.

(Strong)Inductive Hypothesis:
Assume that for all integers less than or equal to k, the statement holds, that is $P(i)$ holds for all $2 \leq i \leq k$.

Inductive Step: Show that $P(k) \rightarrow P(k+1)$

Strong Induction-Example

(Strong)Inductive Step:
Show that $P(k+1)$ is true. More precisely,
$P(i)$ is true (for all $2 \leq i \leq k$) $\rightarrow P(k+1)$

Case (a): Suppose $k+1$ is prime, then $P(k+1)$ is true.

Case (b): Suppose $k+1$ is not a prime, then

$$k+1 = a \times b$$

Clearly, both $2 \leq a$, $b \leq k$

(Note that a and b might not necessarily be k.)

Strong Induction-Example

1. By the (strong) inductive hypothesis, we know that both a and b can be written as the product of primes.
2. Consequently, $k+1$ can be written as the product of primes.
3. Thus, $P(k+1)$ is true, and the statement is true.

Observe that weak inductive hypothesis would not have worked here.

Strong Induction-Example

Example

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

In other words, $P(n)$ = Postage of n cents can be formed using 4-cent and 5-cent stamps. Prove that $P(n)$ is true for all $n \geq 12$.

Strong Induction-Example

Base case

$n = 12, n = 13$, $n = 14, n = 15$

$12 = 4 + 4 + 4$; $13 = 4 + 4 + 5$
$14 = 4 + 5 + 5$; $15 = 5 + 5 + 5$

Strong Inductive Hypothesis

Assume $P(i)$ is true for all $12 \leq i \leq k$.

Inductive Step

Show that $P(k + 1)$ is true.
We can form the postage of $(k + 1)$ cents,
using the stamps that form postage for $(k - 3)$ cents by adding a 4-cent stamp. QED.

Strong Induction -Example

Looking back, do you see why we had to prove the base case for four numbers?

Strong Induction -Example

Looking back, do you see why we had to prove the base case for four numbers?

This was necessary because we make the assumption (in the proof in the inductive step) that the statement is true for $(k - 3)$. If we had only proven that $P(12)$ is true, the base case would not be sufficient to support the claim we make in the induction step.

Total Order and Well-Order

Recall, that a partial order is reflexive, transitive, and anti-symmetric.

A total order is a partial order where every pair of elements are comparable.

And,

A well-order is a total order for which every nonempty subset has a least element.

Well Ordering Principle (WOP)

WOP says that $\mathbb{N}$ is well-ordered. In other words,

Every nonempty subset of nonnegative integers has a smallest element.

- Obvious and straight-forward...
- How can such an "innocent" fact be of any use???

Well, if you believe induction is a powerful technique, then you also believe that WOP is quite powerful. Why? The principle of mathematical induction is logically equivalent to the well-ordering principle.

Well Ordering Principle (WOP)

First, lets see an example of how well ordering principle can be used for proving theorems.

Example Prove that any amount of postage worth 8 cents or more can be made from 3-cent or 5-cent stamps.

(Basically a proof by contradiction that uses WOP)

- Let S be the set of all integers for which the statement is not true.
- S has a smallest element m. Why? Because of the WOP.
- It means that it is not possible to make m cents worth of stamps using only 3-cent and 5-cent stamps.

Proof by Well Ordering Principle (WOP)

We will show that for any m, it is possible to write it as a sum of $3'$s and $5'$s, which will be a contradiction.

For any j in the range, it is possible to make j cents worth of stamps using 3- and 5-cents stamps.

We will show that for any m, it is possible to write it as a sum of 3's and 5's, which will be a contradiction.

- $m = 8$: then $m = 3 + 5$.
- $m = 9$: then $m = 3 + 3 + 3$.
- $m = 10$: then $m = 5 + 5$.
- $m \geq 11$: It means $(m - 3) \geq 8$. Thus, it is possible to write $(m - 3)$ as a sum of $3'$s and $5'$s. Now, add one more 3 to $(m - 3)$ and we get m, which means m can be written as a sum of $3'$s and $5'$s.

Well Ordering Principle (WOP)

Step1 Define the set S of counterexamples, that is
$S = \{i \in Z^+$ such that $P(i)$ is false $\}$

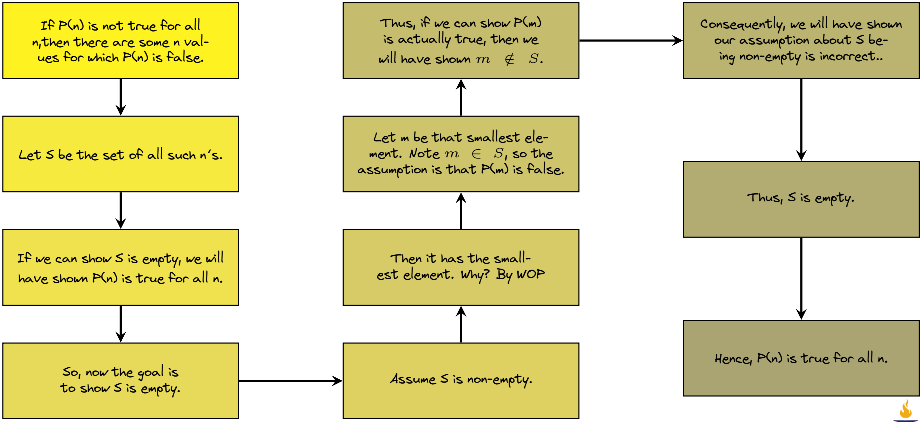
Step2 Assume for contradiction that S is nonempty.

Step3 By the **WOP**, S will have a smallest element m.

Step4 Reach a contradiction somehow – often by showing that $P(m)$ is actually true. (This step requires work)

Step5 Conclude that S must be empty, that is, no counterexample exists.

Prove that $P(n)$ is true for all $n \in \mathbb{Z}^+$ using WOP.



CONCLUSION: Induction