Discrete Structures

Fight for Survival

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August 27, 2026

Substitution of Equivalent Statements

- So far, we have used truth tables to show equivalence between statements.
- Not the easiest way if we have more complex propositions.
- Can there be another way?

Idea:
In a complex statement, substitute parts with equivalent statements until we get the desired statement

It only makes sense to prove some simple but useful equivalences once, and then re-use them whenever they appear in complicated statements.

Substitution of Equivalent Statements

Consider the following statement:

Proposition:
Afrah has a cellphone *AND* she has a laptop.

What is the negation of the above statement? i.e.,

$$\neg(p \wedge q)?$$

Substitution of Equivalent Statements

What is the negation of the following statement:

Proposition:

Afrah has a cellphone **AND** she has a laptop.

- For the above statement to be true, both p and q should be true.
- If p is False or q is False then the above statement becomes False.
- So, the negation of the above statement is:

Negation:

Afrah does not have a cellphone **OR** she does not have a laptop.

$$\neg(p \wedge q) = \neg p \vee \neg q$$

De Morgan's Laws: Logical Equivalence

De Morgan's Laws:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Explanation: De Morgan's Laws provide a way to express the negation of a compound proposition involving logical OR and logical AND in terms of logical AND and logical OR, respectively.

Important Laws of Propositional Logic with some Errors

Law	Equivalence	Equivalence
Commutative Laws	$p \lor q \equiv q \lor p$	$p \land q \equiv q \land p$
Associative Laws	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	$(p \land q) \land r \equiv p \land (q \land r)$
Distributive Laws	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$
De Morgan's Laws	$\neg(p \land q) \equiv \neg p \land \neg q$	$\neg(p \land q) \equiv \neg p \lor \neg q$
Conditional Laws	$p \to q \equiv \neg p \lor q$	$\neg(p \to q) \equiv p \land \neg q$
Biconditional Laws	$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$	
Identity Laws	$p \land F \equiv p$	$p \land T \equiv p$
Domination Laws	$p \lor T \equiv T$	$p \lor F \equiv F$
Double Negation Law	$\neg(\neg p) \equiv \neg p$	
Idempotent Laws	$p \lor p \equiv p$	$p \land p \equiv p$

Important Laws of Propositional Logic

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Idempotent Laws	$p \lor p \equiv p$	$p \land p \equiv p$

Logical Equivalence through Laws

Regarding all these equivalence rules: You should not have to memorize them, instead you should be able to prove them. And know where/when to use them.

Some Important Points:

- Know the Objective
- Be mindful of the order of terms
- Justify each step
- There could be many ways

So, what's the trick here?

- Practice, and
- Some more practice
- Justify each step
- Keep in mind the objective.

Example: $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

Prove: $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

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Using De Morgan's Law:

$$\begin{aligned} \neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) \\ &\equiv \neg p \wedge (p \vee \neg q) \end{aligned}$$

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Using Distributive Law:

$$\begin{aligned}\neg p \wedge (p \vee \neg q) &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \\ &\equiv F \vee (\neg p \wedge \neg q) \\ &\equiv \neg p \wedge \neg q\end{aligned}$$

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Thus, $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

Summary of Propositional Logic so far

- Introduced basic concepts of propositional logic: propositions, logical operators, truth values.
- Explored logical operators: conjunction ($\wedge$), disjunction ($\vee$), negation ($\neg$), conditional ($\to$), biconditional ($\leftrightarrow$).
- Demonstrated truth tables to evaluate compound propositions.
- Discussed logical equivalence and its importance.
- Covered De Morgan's Laws for simplifying expressions.
- Explored techniques for working with complex statements, including substitution.

Keep in mind: Logic provides tools for reasoning and problem-solving in various domains.