Discrete Structures

Fight for Survival

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August 27, 2026

PIGEONHOLE PRINCIPLE

Counting: Using Set Properties

Recall: We can compare the cardinality of two sets by properties of functions:

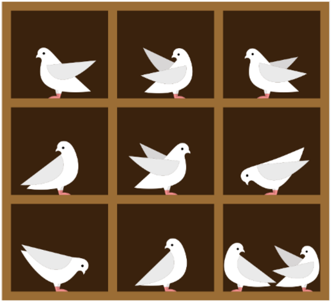
- A surjection from set A to set B, then $|A| \geq |B|$
- An injection from set A to set B, then $|A| \leq |B|$
- A bijection from set A to set B, then $|A| = |B|$

The bijection rule says that if there is a bijection from one set to another then the two sets have the same cardinality.

Pigeonhole Principle

If A and B are finite sets with $|A| > |B|$, then there are no injective functions from A to B. This is called the pigeonhole principle.

If m things are to be put into n places and m > n, then (at least) one place has two or more things in it.



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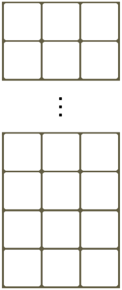
In general,

If $n \geq km + 1$ objects are distributed among m sets, then at least one of the sets will contain at least $(k+1)$ objects.

What is the smallest number of people, so that at least two of them will have same birthday (month and date)?

Pigeonhole Principle

Consider a grid of (4×82) points such that each point is colored either red, green, or blue. Prove that there always exists a rectangle in the grid such that all four of its vertices are the same colors.



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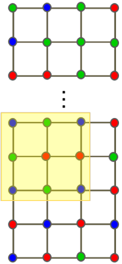

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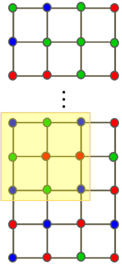
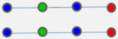


- Each row has 4 points that need to be colored using 3 colors.
- If there are at least two rows whose points are colored the same way, we know the required rectangle exists. 
- Each row can be colored in one of the $3^4 = 81$ ways.
- If we have more than 81 rows, two rows will always exist whose points have same colors. Why?
- By the Pigeonhole principle. Hence, the desired result follows.

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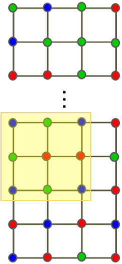

Pigeon holes: Coloring Options
Pigeons: Rows of the Grid

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In a group of 680 students, is it true that there are always two that have the same first and last initial?

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YES!

- There are $26 \times 26 = 676$ possible combinations of initials.
- So in the worst case, you would go through 676 people without finding a matched set of initials.
- But you will find a match with the 677th person's initials.

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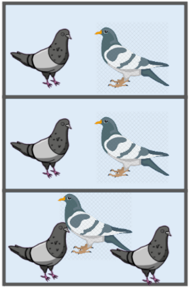
Pigeon holes: Letter combinations for initials

Pigeons: Students (680)

The Generalized Pigeonhole Principle

Consider a function whose domain has n elements and whose target has k elements, for n and k positive integers. Then there is an element y in the target such that **f maps at least $\lceil n/k \rceil$ elements in the domain to y.**

Example:



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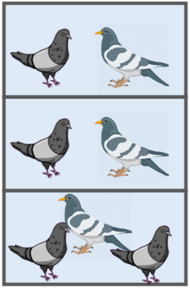

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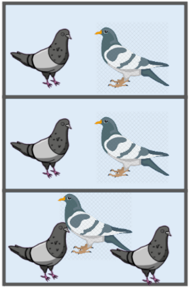
Example:

- $n = 7$
- $k = 3$
- $\lceil n/k \rceil = 3$



The Generalized Pigeonhole Principle

There are 121.4 million people in the United States who earn an annual income that is at least \$10,000 and less than \$1,000,000. Show that there are 123 people who earn the same annual income in dollars.

Answer:

(Total population) n = 121,400,000

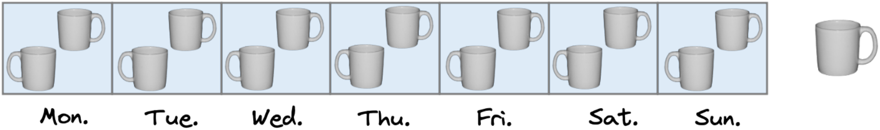
(No. of possible incomes) k = 990,000

$\lceil n/k \rceil$ = 123

Therefore, there are at least 123 people (pigeons) that have the same income (box)!

The Generalized Pigeonhole Principle

Afrah finished 15 cups of coffee in a week. Show that there must be a day when Afrah drank at least three cups.



(Total coffee cups) $n = 15$
(No. of weekdays) $k = 7$
$\lceil n/k \rceil = 3$