Discrete Structures
Fight for Survival

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Relations - Partial Order

Let's consider the following definition:

Partial Order:
A relation R on a set A is called a partial order or partial ordering if it is reflexive, anti-symmetric, and transitive at the same time.

1. Reflexive $[\forall a \in A, (a, a) \in R]$
2. Anti-symmetric $[(a, b) \wedge bRa \implies bRa]$
3. Transitive $[aRb \wedge bRc \implies aRc]$

Example: The greater than or equal to ($\geq$) relation is a partial ordering on the set of integers (Z).

Question: HOW?

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Example: The greater than or equal to $(\geq)$ relation is a partial ordering on the set of integers (Z).
Question:HOW?

- As $a \geq a$, for every integer a. Hence $\geq$ is reflexive.
- If $(a \geq b)$ and $(b \geq a)$ then, $a = b$. Hence $\geq$ is anti-symmetric.
- If $(a \geq b)$ and $(b \geq c)$, then $(a \geq c)$. Hence $\geq$ is transitive.

More terminology on Partial Order

If a relation R is partial order on set A, then the set A is called partially ordered set, or poset and represented by (A, R). Members of A are called elements of the poset.

More terminology on Partial Order

Two elements of a partially ordered set, x and y, are said to be comparable if $x \preceq y$ or $y \preceq x$. Otherwise they are said to be incomparable.

Example

In the poset $(\mathbb{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution : The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$

When every two elements in the set S are comparable, the relation is called a total ordering and S is called a totally ordered or Linearly ordered set

Minimal and Maximal Elements

An element x is a minimal element in the POSET if there is no $y \neq x$ such that $y \preceq x$.

An element x is a maximal element in the POSET if there is no $y \neq x$ such that $x \preceq y$.

Maximal and minimal elements are easy to spot using a Hasse diagram. They are the "top" and "bottom" elements in the diagram. There can be multiple maximal or minimal elements in S.

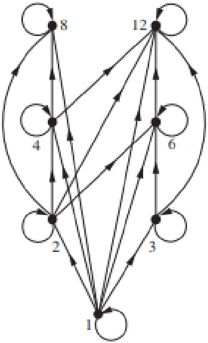
Hasse Diagram

A Hasse diagram is a graphical representation of the relation of elements of a partially ordered set (POSET) with an implied upward orientation. While drawing a Hasse diagram following points must be remembered :

1. Since a partial order is reflexive, hence each vertex of A must be related to itself, so the edges from a vertex to itself are deleted in Hasse diagram.
2. Since a partial order is transitive, hence whenever aRb, bRc, we have aRc. Delete edge from a to c but retain the other two edges.
3. If a vertex 'a' is connected to vertex 'b' by an edge, i.e., $a < b$, then the vertex 'b' appears above vertex 'a'. Therefore, the arrow may be omitted from the edges in the Hasse diagram.

Hasse Diagram

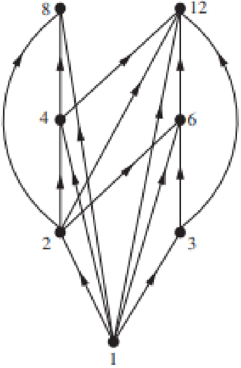
Constructing the Hasse Diagram of $(\{1, 2, 3, 4, 6, 8, 12\}, |)$.



(a) Directed Graph

Hasse Diagram

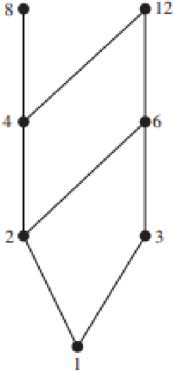
Constructing the Hasse Diagram of (1, 2, 3, 4, 6, 8, 12, |).



(b) Remove self loops

Hasse Diagram

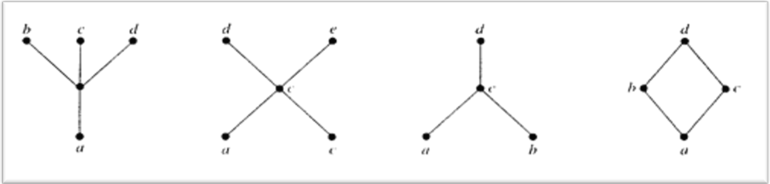
Constructing the Hasse Diagram of (1, 2, 3, 4, 6, 8, 12, |).



(c) Remove transitivity and directions

Greatest and Least Elements

A maximal element $x \in S$ is the greatest element of S if $y \preceq x$ (a successor) for all $y \in S$.
A maximal element $x \in S$ is the least element of S if $x \preceq y$ (a predecessor) for all $y \in S$.



Greatest elements : None, None, d, d
Least elements: a, None, None, a

Sets and Subsets

Consider the following problem:

Let the sets $A_i, 1 \leq i \leq k$, be distinct subsets of $\{1, 2, \ldots, n\}$. Suppose $A_i \cap A_i \neq \emptyset$ for all i and j. Show that $k \leq 2^{n-1}$ and give an example where equality holds.