Discrete Structures

Fight for Survival

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Quantifiers

A predicate can become a proposition when we assign it a fixed value from the universe of discourse.

But there's another powerful tool: quantification. This allows us to determine whether the predicate is true (or false) for all possible values in the universe of discourse or for some specific value(s).

- Universal Quantifier ($\forall$): True or false for all values in the universe of discourse.
- Existential Quantifier ($\exists$): True or false for some values in the universe of discourse.

Universal Quantification: Examples

Universal quantification ($\forall$) is like a magic spell that makes statements apply to *every* member of a set.

Let's see some fun examples:

- $\forall$ **Cats**: Love napping in sunbeams.
- $\forall$ **Programmers**: Drink coffee by the gallon.
- $\forall$ **Professors**: Claim their subject is the most interesting.

Universal Quantifier - Example

Let P(x)= "x must take the Discrete Structure course",
and Q(x)= "x is a Computer Science student".
The universe of discourse is "all students of ITU (x)".

- Express the statement in symbolic form: "All computer science students must take the Discrete Structure course"

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$$\forall x(Q(x) \rightarrow P(x))$$

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Existential Quantifier

Definition

The existential quantification of a predicate P(x) is the proposition "P(x) is true for some of the values of x in the universe of discourse". We use the notation

$$\exists x P(x)$$

which is read as "there exists x".

Similarly, if the universe of discourse is finite, say $\{n_1, n_2, n_3, ..., n_k\}$, then the existential quantifier is simply the disjunction of all the elements

$$\exists x P(x) \equiv P(n_1) \vee P(n_2) \vee P(n_3) \vee ... \vee P(n_k)$$

Existential Quantification: Examples

Existential quantification (∃) allows us to find at least one instance where a statement holds true.

Let's explore some practical examples:

- ∃ Students Who Love Math: In a class of 60, some are passionate about math.
- ∃ Available Parking Spots: In a busy Arfa Tower, there are spots waiting to be found.

Truth values of Quantifiers

In general, when are quantified statements true/false?

Statement	True When	False When
$\forall x P(x)$	P(x) is true for every x	There is an x for which P(x) is false
$\exists x P(x)$	There is an x for which P(x) is true	P(x) is false for every x

Table: Truth Conditions of Universal and Existential Quantifiers

Example - Express in Mathematical Notation

- The product of two negative integers is positive.
- The difference of two negative integers is not necessarily negative.

Example - Express in Mathematical Notation

- The product of two negative integers is positive.

$$\forall x, y \in \mathbb{Z}^-, x \times y > 0$$

- The difference of two negative integers is not necessarily negative.

$$\exists a, b \in \mathbb{Z}^-, a - b > 0$$

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the other logical operators.
For example,

$$\exists a P(a) \lor Q(a) \equiv (\exists a P(a)) \lor Q(a)$$

$$\exists a P(a) \lor Q(a) \not\equiv \exists a (P(a) \lor Q(a))$$

Mindful of the Domain

Always be mindful of the domain when dealing with quantifiers. The same statement can be true or false based on the domain.

For example:

- $\forall x(x^2 > x)$
  - Domain: x is a positive integer
  - This statement is false.
  - Counterexample: $x = 1$
- $\forall x(x^2 > x)$
  - Domain: x is a positive integer greater than 1
  - This statement is true.

Proving/Disproving Quantifiers

To prove universally quantified statements, we need to show that the statement is true for all possible values in the domain.

Proving/Disproving Quantifiers

To disprove universally quantified statements, we need to show that the statement is false for just one value in the domain. (also referred to as the counterexample).

Proving/Disproving Quantifiers

To prove existentially quantified statements, we need to show that the statement is true for just one value in the domain.

Proving/Disproving Quantifiers

To disprove existentially quantified statements, we need to show that the statement is false for every value in the domain.

A question:

If a universally quantified statement is true, does that mean the existentially quantified statement is always true?

Free and Bound variables

- A variable x in the predicate P(x) is called a free variable because the variable is free to take on any value in the domain.
- The variable x in the statement $\forall x P(x)$ or $\exists x P(x)$ is a bound variable because the variable is bound to the quantifier within the domain.
- A statement with no free variables is a proposition because the statement's truth value can be determined.
- The part of logical expression to which a quantifier is applied is called the scope of the quantifier.

Free and Bound variables

- $\forall x P(x) \land Q(x)$ is not a proposition.
- But $\forall x (P(x) \land Q(x))$ is a proposition.

Quantifiers - Negation (DeMorgan's Law)

$\forall x P(x)$: Every student in this class has taken a calculus course.

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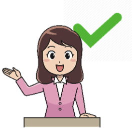

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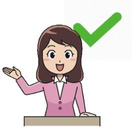

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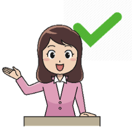

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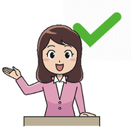
 and  and 

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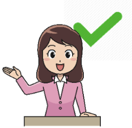
 **and**  **and** 

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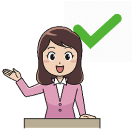
 **and**  **and** 

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$$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$$

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Domain of discourse $\{x_1, x_2, \dots, x_n\}$

$\neg \forall x P(x)$

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$$\neg\forall x P(x) \equiv \neg(P(x_1) \land P(x_2) \land P(x_3) \land \ldots \land P(x_n))$$

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$\neg \forall x P(x) \equiv \neg(P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \ldots \wedge P(x_n))$

$\equiv$

$\neg P(x_1) \vee \neg P(x_2) \vee \neg P(x_3) \vee \ldots \vee \neg P(x_n)$

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$$\equiv$$

$$\exists x \neg P(x) \equiv \neg P(x_1) \lor \neg P(x_2) \lor \neg P(x_3) \lor \ldots \lor \neg P(x_n)$$

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$\neg \exists x P(x)$ $\equiv$ $\neg(P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots \vee P(x_n))$

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$$\neg \exists x P(x) \equiv \neg(P(x_1) \vee P(x_2) \vee P(x_3) \vee \ldots \vee P(x_n))$$

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