Discrete Structures

Fight for Survival

Dr. Mudassir Shabbir

August 27, 2026

Relations

Recall the Cartesian Product of two sets, $A = \{1, 2, 3, 4\}$ and $Z = \{A, B, C\}$ is,
$A \times Z = \{(1, A), (1, B), (1, C),\ (2, A), (2, B), (2, C),\ (3, A), (3, B), (3, C), (4, A), (4, B), (4, C)\}$

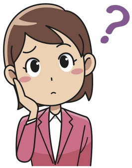
Relation:
Let A and Z be two sets. A binary relation R from A to Z is a subset of the Cartesian Product $A \times Z$, i.e., R ⊆A×Z

. Examples: ,
$R = \{(2, A), (3, B), (4, C)\}$
$R = \{\}$
$R = \{(2, A)\}$

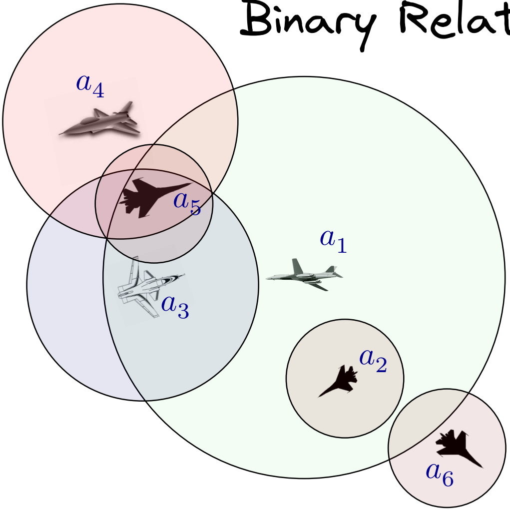
Relations? Hang on a Sec!

- I know Sets.
- I know Cartesian Product of Sets.
- I also know subsets and their applications.

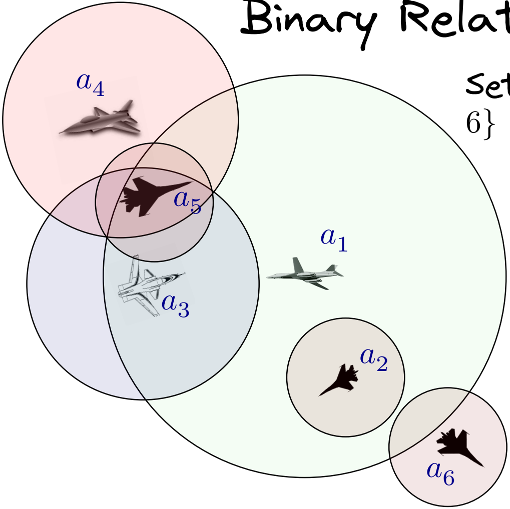
But what the heck is a Subset of the Cartesian Product of a Set useful for?



Binary Relations

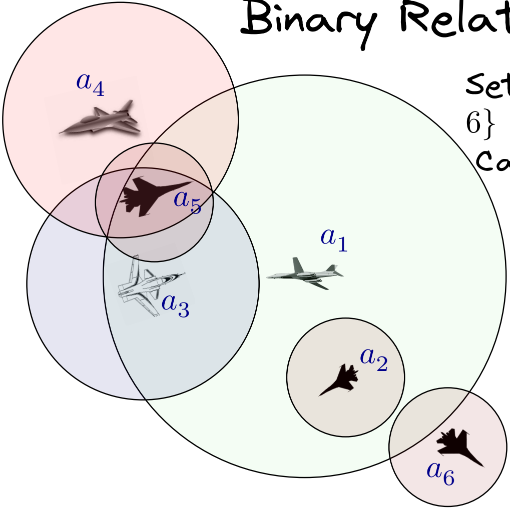


Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

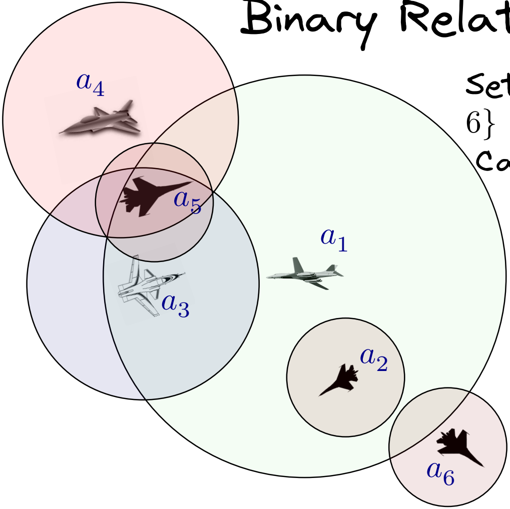
Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

Cartesian Product = M$\times$M

Binary Relations

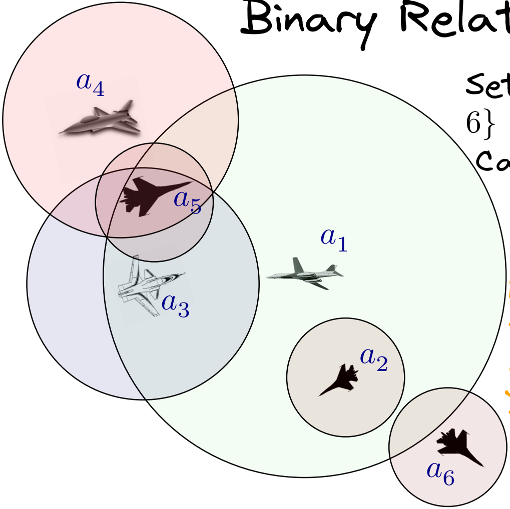


Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

Cartesian Product = $M \times M$

Relation: Who can see who?

Binary Relations



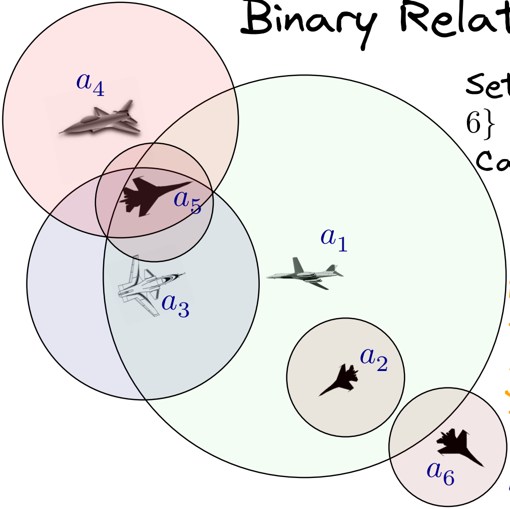
Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

Cartesian Product = M $\times$ M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

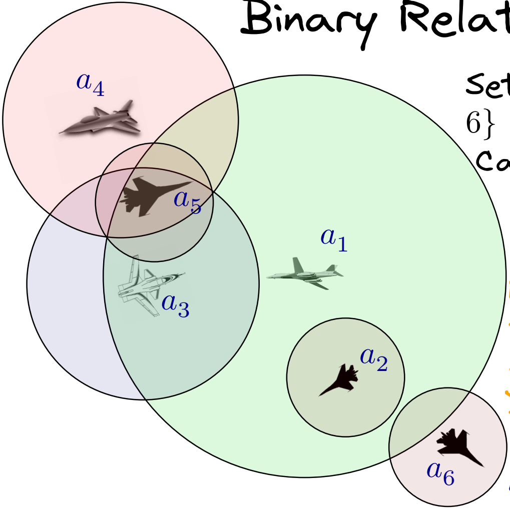
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

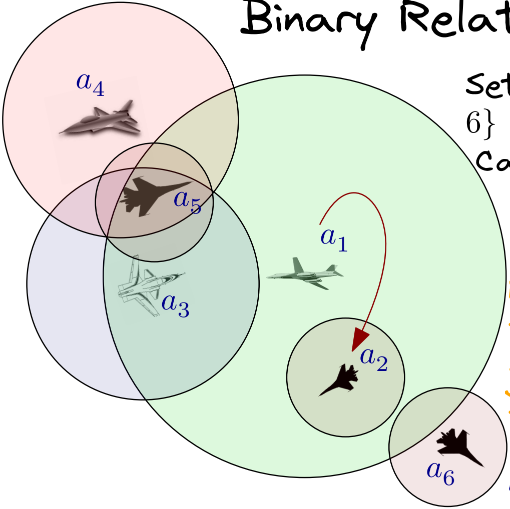
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

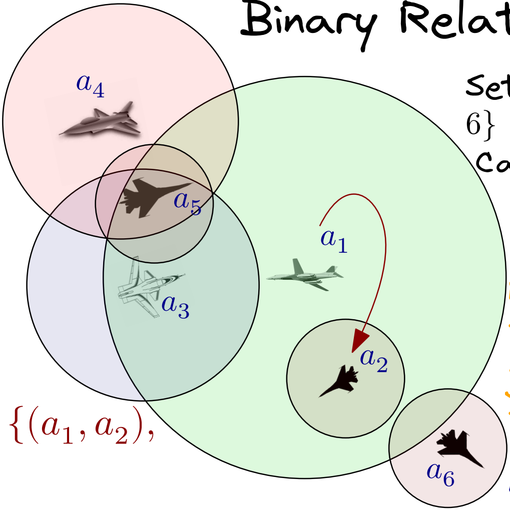
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

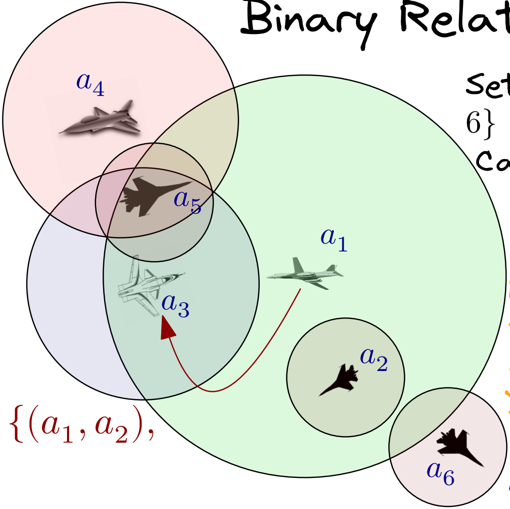
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



$\{(a_1, a_2),$

Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

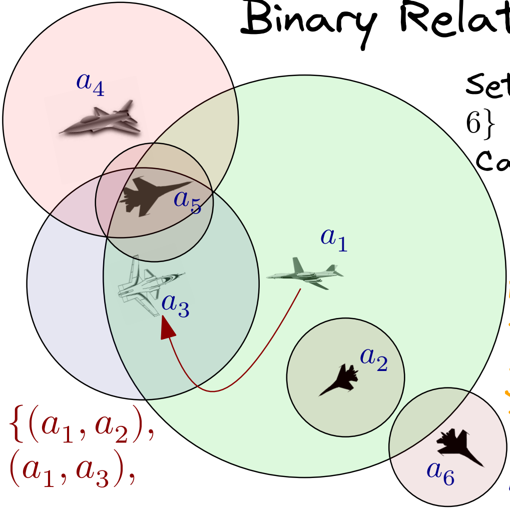
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



$\{(a_1, a_2),$
$(a_1, a_3),$

Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

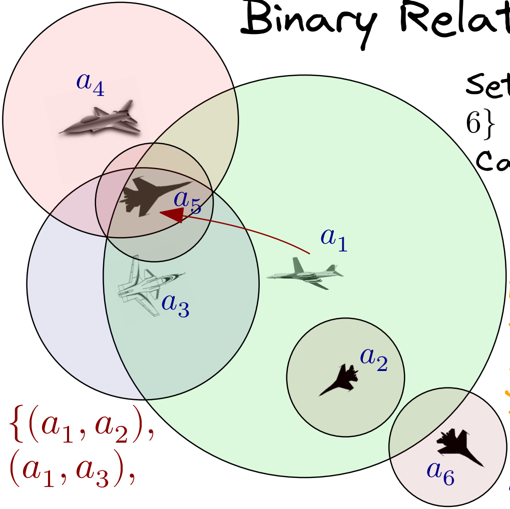
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



$\{(a_1, a_2),$
$(a_1, a_3),$

Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

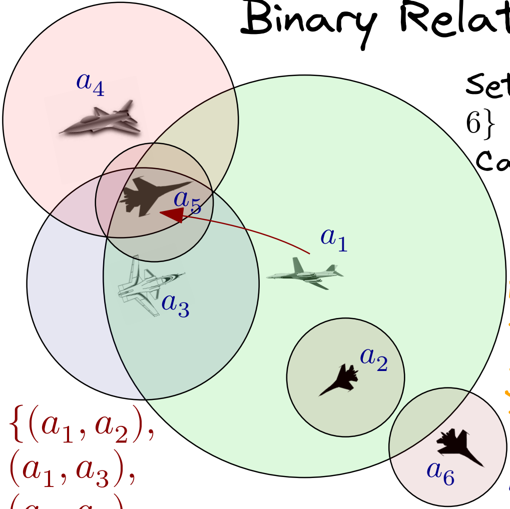
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

Cartesian Product = M$\times$M

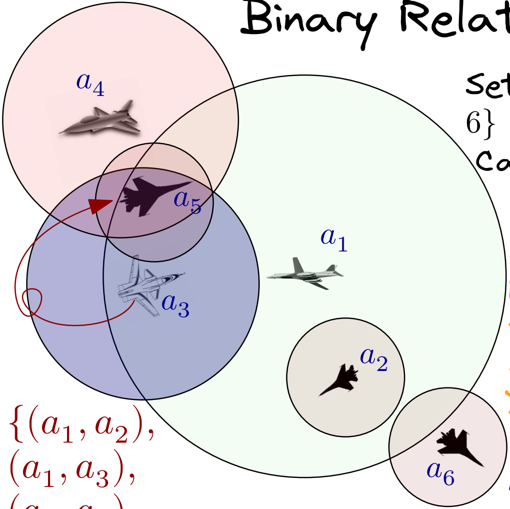
Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

$\{(a_1, a_2),$
$(a_1, a_3),$
$(a_1, a_5),$

Note: A can see B does NOT mean B can also see A!

Binary Relations



$\{(a_1, a_2),$
$(a_1, a_3),$
$(a_1, a_5),$

Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

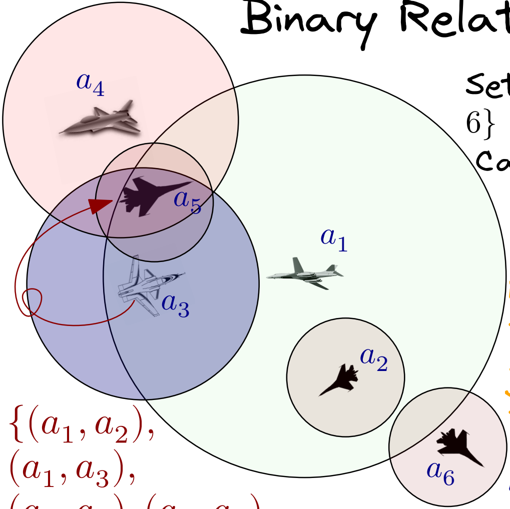
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



$\{(a_1, a_2),$
$(a_1, a_3),$
$(a_1, a_5), (a_3, a_5),$

Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

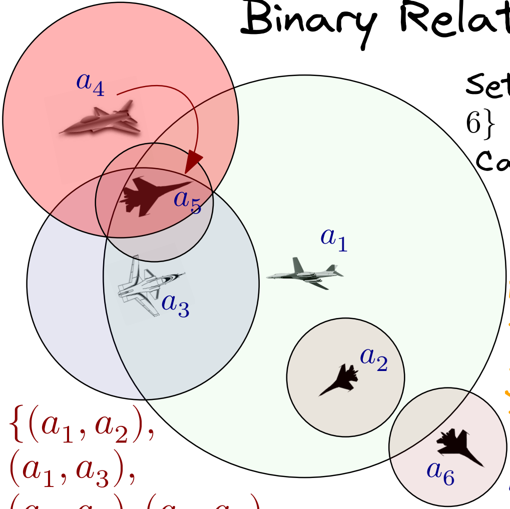
Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

Note: A can see B does NOT mean B can also see A!

Binary Relations



Set of jets M = $\{a_i | 1 \le i \le 6\}$

Cartesian Product = M×M

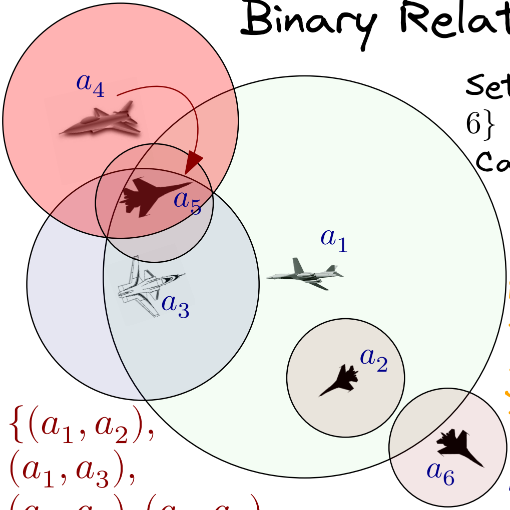
Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

$\{(a_1, a_2),$
$(a_1, a_3),$
$(a_1, a_5), (a_3, a_5),$

Note: A can see B does NOT mean B can also see A!

Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

Cartesian Product = M$\times$M

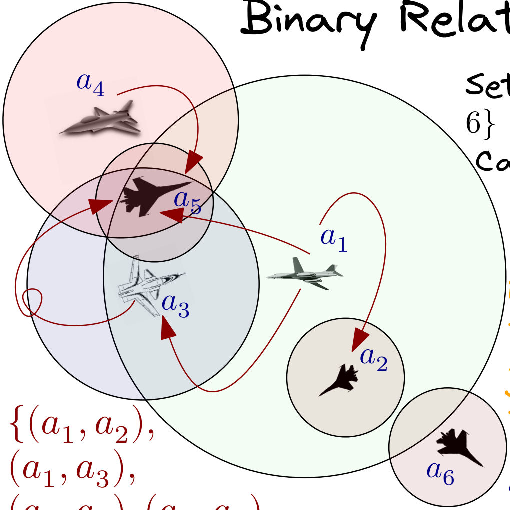
Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

$\{(a_1, a_2), (a_1, a_3), (a_1, a_5), (a_3, a_5), (a_4, a_5)\}$

Note: A can see B does NOT mean B can also see A!

Binary Relations



Set of jets M = $\{a_i | 1 \le i \le 6\}$

Cartesian Product = M$\times$M

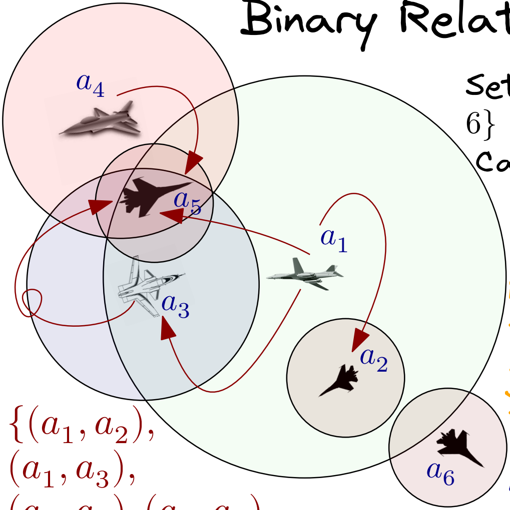
Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

$\{(a_1, a_2),$
$(a_1, a_3),$
$(a_1, a_5), (a_3, a_5),$
$(a_4, a_5)\}$

Note: A can see B does NOT mean B can also see A!

Binary Relations



Set of jets M = $\{a_i | 1 \leq i \leq 6\}$

Cartesian Product = M$\times$M

Relation: Who can see who?

Each jet has a sensing circle and can see all jets that lie in its sensing circle.

$\{(a_1, a_2), (a_1, a_3), (a_1, a_5), (a_3, a_5), (a_4, a_5)\} \subseteq$ M$\times$M

Note: A can see B does NOT mean B can also see A!

Relations

Recall the Cartesian Product of two sets, $A = \{1, 2, 3, 4\}$ and $Z = \{A, B, C\}$ is,
$A \times Z = \{(1, A), (1, B), (1, C),\ (2, A), (2, B), (2, C),\ (3, A), (3, B), (3, C), (4, A), (4, B), (4, C)\}$

Relation:
Let A and Z be two sets. A binary relation R from A to Z is a subset of $A \times Z$, i.e., R $\subseteq$A×Z

Examples: ,
$R = \{(2, A), (3, B), (4, C)\}$
$R = \{\}$
$R = \{(2, A)\}$

Relations - Example

Let $A =$ Set of cities ; $B =$ Set of countries

$(a, b) \in R$ where city a is the capital of country b.
(Riyadh, Saudi Arabia), (Delhi, India), (Washington, USA) are in R.

$$R = \{(a, b) : a|b \wedge a, b \in \mathbb{Z}\}$$

$$R = \{(a, b, c) : a^2 = b^2 + c^2 \wedge a, b \in \mathbb{Z}\}$$

Relations- Example

A binary relation R on set A is a subset of $A \times A$

Say, $A = \{0, 1, 2, 3\}$, we can have examples;

- The relation is $=$
$$\{(0,0),(1,1),(2,2),(3,3)\}$$
- The relation is $<$
$$\{(0,1),(0,2),(0,3),(1,2),(1,2),(2,3)\}$$
- The relation is $\geq$
$$\{(0,0),(1,1),(1,0),(2,2),(2,1),(2,0),(3,3),(3,2),(3,1),(3,0)\}$$

Relations - Example

Consider the following relations defined on the set of reals and the set of integers (that are infinite)

From reals to integers, xRy if $|x-y| \leq 1$

In other words, xRy (x is related to y) if the distance between real number x and integer y is at most 1.

$$R_1 = \{(a,b) : a = b \wedge a, b \in \mathbb{R}\}$$

This relation is a straight line in 2 dimensions.

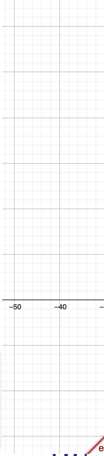
$$R_2 = \{(a,b,c) : a^2 + b^2 + c^2 \leq 1 \wedge a, b, c \in \mathbb{R}\}$$

This relation is a ball of radius 1 in three dimensions.

$$R_3 = \{(a,b) : a \in \mathbb{Z} \wedge b \in \mathbb{R}\}$$

This relation is a infinite set of vertical lines.

Relations - Examples



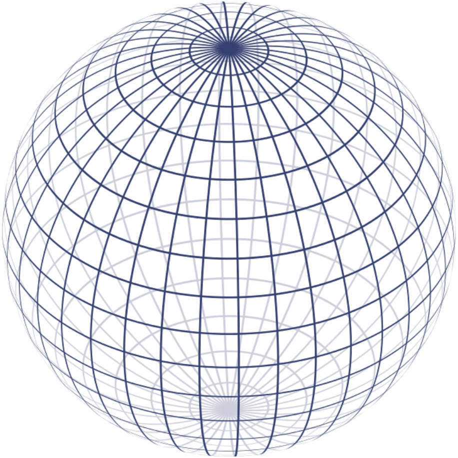
$$R_1 = \{(a,b) : a = b \wedge a, b \in \mathbb{R}\}$$

This relation is a straight line in 2 dimensions.

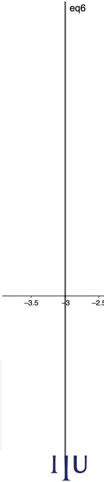
Relations - Examples

$$R_2 = \{(a, b, c) : a^2 + b^2 + c^2 \leq 1 \wedge a, b, c \in \mathbb{R}\}$$

This relation is a ball of radius 1 in three dimensions.



Relations - Examples



$$R_3 = \{(a, b) : a \in \mathbb{Z} \land b \in \mathbb{R}\}$$

This relation is a infinite set of vertical lines.

Relations - On a set

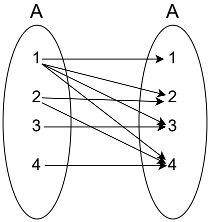
A relation on a set A is a mapping from $A \rightarrow A$. As defined, a relation on a set A is a subset of $A \times A$.
Let, $A = \{1, 2, 3, 4\}$. Which ordered pairs are there in the following relation?

$$R = \{(a, b) | a | b\}$$

Solution: $(a, b) \in R$ if and only if $a, b \in A$ and a divides b, we get:

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

We can also represent it like this:



Relations - On infinite set

Consider the following (infinite) relations on the set of integers Z

$R_1 = \{(a, b)|a < b\}$

$R_2 = \{(a, b)|a > b\}$

$R_3 = \{(a, b)|a = b\}$

$R_4 = \{(a, b)|a = b \text{ or } a = -b\}$

$R_5 = \{(a, b)|a = b + 1\}$

$R_6 = \{(a, b)|a + b \leq 3\}$

Question: Which of the above relations contains ordered pairs $(1, 1), (1, 2), (2, 1), (1, -1)$, and $(2, 2)$,

Relations - On infinite set

Consider the following (infinite) relations on the set of integers Z

$R_1 = \{(a, b) | a < b\}$

$R_2 = \{(a, b) | a > b\}$

$R_3 = \{(a, b) | a = b\}$

$R_4 = \{(a, b) | a = b \text{ or } a = -b\}$

$R_5 = \{(a, b) | a = b + 1\}$

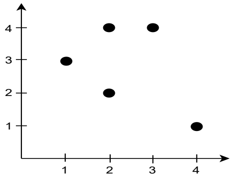
$R_6 = \{(a, b) | a + b \leq 3\}$

Question: Which of the above relations contains ordered pairs $(1, 1), (1, 2), (2, 1), (1, -1)$, and $(2, 2)$

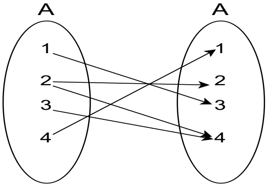
Answer: The pair $(1, 1)$ is in R_3, R_4 and R_6 ; $(1, 2)$ is in R_1 and R_6; $(2, 1)$ is inR_2, R_5 and R_6 ; $(1, -1)$ is in R_2, R_3 and R_6 ; and finally, $(2, 2)$ is in R_1, R_3 and R_4.

Relation - Representation

For $R = \{(1,3),(2,2),(2,4),(3,4),(4,1)\}$ on the set $A = \{1,2,3,4\}$, we can represent the R as:



Coordinate Diagram

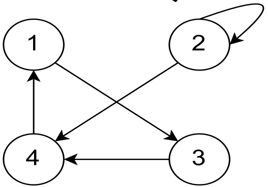


Arrow Diagram

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Matrix Representation



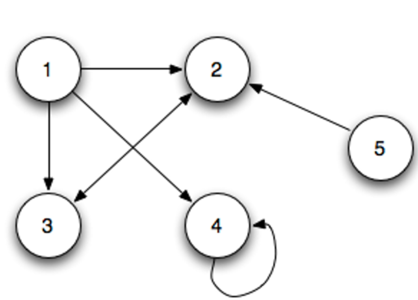
Directed Graph

Relations - Arrow and Matrix Representations

Arrow diagram or Directed Graph: For a binary relation from A to B, there is an arrow from $a \in A$ to $b \in B$ if aRb.

A **matrix representation** is a rectangular array of numbers with $|A|$ rows and $|B|$ columns.

- Each row corresponds to an element of A.
- Each column corresponds to an element of B.
- There is a 1 in row a, column b, if aRb. Otherwise, there is a 0.



	1	2	3	4	5
1	0	1	1	1	0
2	0	0	1	0	0
3	0	1	0	0	0
4	0	0	0	1	0
5	0	1	0	0	0

Relations - Properties

The relation R on a set A is reflexive if and only if

$$xRx \text{ for every } x \in A$$

In other words, every x is related to itself.

Arrow Diagrams of Reflexive relations.

Relations - Properties

Which of the following relations are reflexive?

Example: Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (4,4), (2,2), (3,3)\}$

$R_2 = \{(1,1), (1,4), (2,2), (3,3), (4,2)\}$

$R_3 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$

$R_4 = \{(1,3), (2,2), (3,1), (2,4), (3,3), (4,4)\}$

Relations - Properties

Which of the following relations are reflexive?

Example: Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1,1), (4,4), (2,2), (3,3)\}$ Reflexive

$R_2 = \{(1,1), (1,4), (2,2), (3,3), (4,2)\}$ Non-Reflexive

$R_3 = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4)\}$ Reflexive

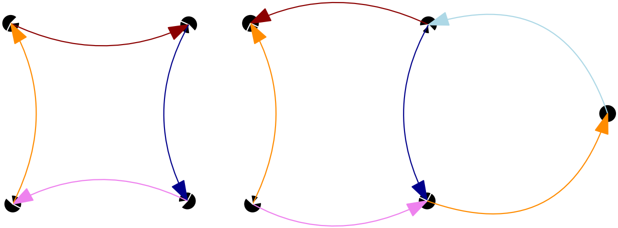
$R_4 = \{(1,3), (2,2), (3,1), (2,4), (3,3), (4,4)\}$ Non-Reflexive

Reflexive relation is a relation of elements of a set A such that each element of the set is related to itself.

Relations - Properties

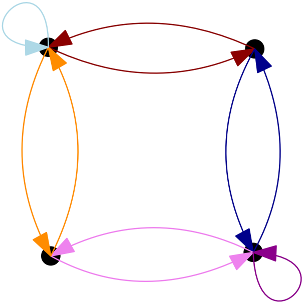
The relation R on a set A is anti-reflexive or ir-reflexive if and only if

$$(x, x) \notin R \text{ for every } x \in A$$



Arrow Diagrams of Anti-Reflexive relations.

Relations - Properties



Note: it is quite possible to have a relation that is neither reflexive nor anti-reflexive.

Relations - Properties

The relation R on a set A is Symmetric if and only if

$$(x, y) \in R \implies (y, x) \in R \text{ for every } x, y \in A$$

Arrow Diagrams of Symmetric relations.

Relations - Properties

Let's consider the following definition:

Symmetry:
A relation R on a set A is called symmetric if $(b, a) \in R$, whenever $(a, b) \in R$.

Example: Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (4, 4), (2, 2), (3, 3)\}$

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 2)\}$

$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

$R_4 = \{(1, 3), (2, 2), (3, 1), (2, 4), (3, 3), (4, 4)\}$

Relations - Properties

Let's consider the following definition:

Symmetry:
A relation R on a set A is called symmetric if $(b, a) \in R$, whenever $(a, b) \in R$.

Example: Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (4, 4), (2, 2), (3, 3)\}$ Symmetric

$R_2 = \{(1, 1), (1, 4), (2, 2), (3, 3), (4, 2)\}$ Non-Symmetric (asymmetric)

$R_3 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$ Symmetric

$R_4 = \{(1, 3), (2, 2), (3, 1), (2, 4), (3, 3), (4, 4)\}$ Non-Symmetric (asymmetric)

Relations - Properties

The relation R on a set A is Anti-symmetric if and only if

$$(x, y) \in R \wedge (y, x) \in R \implies x = y \text{ for every } x, y \in A$$

Arrow Diagrams of Anti-Symmetric relations.

Relations - Properties

The relation R on a set A is Transitive if and only if

$$(x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R \text{ for every } x, y, z \in A$$

Arrow Diagrams of Transitive relation.

Relations - Properties

The relation R on a set A is Transitive if and only if

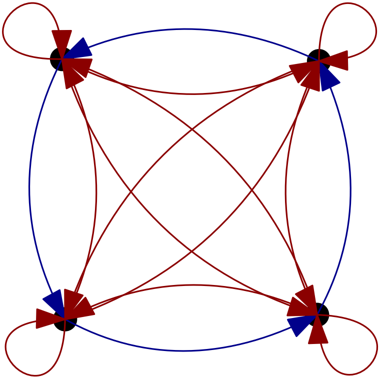
$$(x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R \text{ for every } x, y, z \in A$$

Arrow Diagrams of Transitive relation.

Relations - Properties

The relation R on a set A is Transitive if and only if

$$(x,y) \in R \wedge (y,z) \in R \implies (x,z) \in R \text{ for every } x,y,z \in A$$



Arrow Diagrams of Transitive relation.

Relations - Properties

Let's consider the following definition:

Transitivity:
A relation R on a set A is called Transitive if $(a, b) \in R$ and $(b, c) \in R$, imply that $(a, c) \in R$, where all $a, b, c \in A$.

Example: Let $A = \{1, 2, 3, 4\}$

$$R_1 = \{(1,1), (1,2), (1,3), (2,3)\}$$

$$R_2 = \{(1,2), (1,3), (2,2), (3,3), (2,4)\}$$

$$R_3 = \{(1,1), (2,2), (4,4), (3,3), (4,3)\}$$

$$R_4 = \{(1,3), (2,2), (2,4), (3,4), (4,1)\}$$

Relations - Properties

Let's consider the following definition:

Transitivity:
A relation R on a set A is called Transitive if $(a, b) \in R$ and $(b, c) \in R$, imply that $(a, c) \in R$, where all $a, b, c \in A$.

Example: Let $A = \{1, 2, 3, 4\}$

$R_1 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$ Transitive

$R_2 = \{(1, 2), (1, 3), (2, 2), (3, 3), (2, 4)\}$ Non-Transitive

$R_3 = \{(1, 1), (2, 2), (4, 4), (3, 3), (4, 3)\}$ Transitive

$R_4 = \{(1, 3), (2, 2), (2, 4), (3, 4), (4, 1)\}$ Non-Transitive

Relations - Properties

The relation R on a set A is Anti-Transitive if and only if

$$(x, y) \in R \wedge (y, z) \in R \implies (x, z) \notin R \text{ for every } x, y, z \in A$$

Arrow Diagrams of Anti-transitive relation.

The relation R on a set A is Anti-Transitive if and only if

$$(x, y) \in R \land (y, z) \in R \implies (x, z) \notin R \text{ for every } x, y, z \in A$$

Arrow Diagrams of Anti-transitive relation.

Relations Properties - Practice

Consider the relation,

R: less than or equal over the set of real numbers.

$$xRy \equiv x \leq y$$

Explain your reasoning.

1. Is R reflexive?
2. Is R symmetric?
3. Is R transitive?
4. Is R anti/irreflexive?
5. Is R anti-symmetric?

Relations Properties - Practice

Is it reflexive?

(xRx for all $x \in A$)?

Relations Properties - Practice

Is it reflexive?

(xRx for all $x \in A$)?

- Yes!

- If $x \in A$ then $x \leq x$ is also true since they are equal. So the relationship is reflexive (consider $5 \leq 5$).

Relations Properties - Practice

Is it symmetric?

($xRy \implies yRx$ for all $x, y \in A$)?

Relations Properties - Practice

Is it symmetric?

$(xRy \implies yRx$ for all $x, y \in A)$?

- No.

- $(5,6) \in R$ but $(6,5) \notin R$.

Relations Properties - Practice

Is it Anti-Reflexive?

(xRx is not true for all $x \in A$)?

Relations Properties - Practice

Is it Anti-Reflexive?

(xRx is not true for all $x \in A$)?

- No.
- Counterexample: $5R5$.

Relations Properties - Practice

Is it Anti-Symmetric?

(xRy and yRx implies $x = y$ for all $x, y \in A$)?

Relations Properties - Practice

Is it Anti-Symmetric?

(xRy and yRx implies $x = y$ for all $x, y \in A$)?

- YES.

- The only way $x \leq y$ and at the same time $y \leq x$ is when $x = y$.

A Question

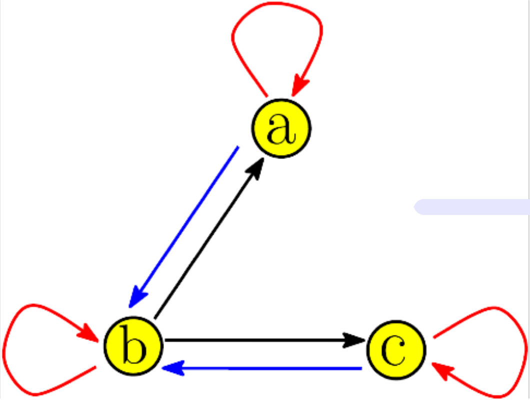
True or False?

If a binary relation is transitive and symmetric, then it is also reflexive?

A Question

True or False?

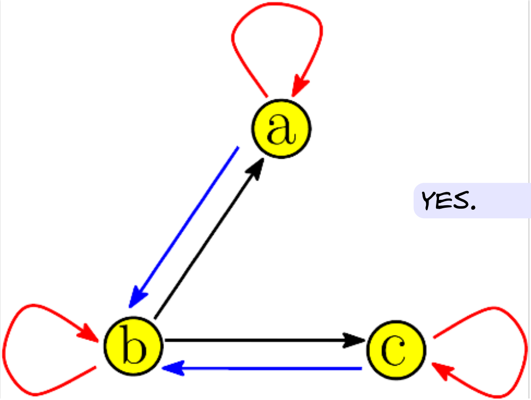
If a binary relation is transitive and symmetric, then it is also reflexive?



A Question

True or False?

> If a binary relation is transitive and symmetric, then it is also reflexive?

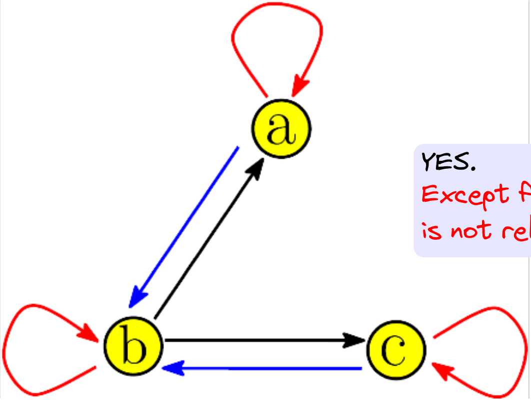


YES.

A Question

True or False?

If a binary relation is transitive and symmetric, then it is also reflexive?



YES.
Except for a trivial case when an element is not related to any other element.

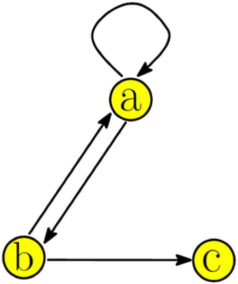
Closure of Relation

The closure of a relation R with respect to property P is the relation obtained by adding the minimum number of ordered pairs to R to obtain the property P.
Lets see the closure of relations with respect to the following properties:

- Reflexive closure
- Symmetric closure
- Transitive closure

Reflexive Closure

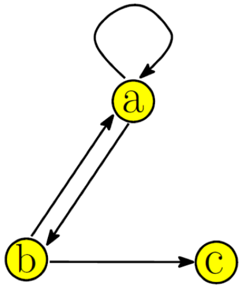
Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$



Reflexive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$

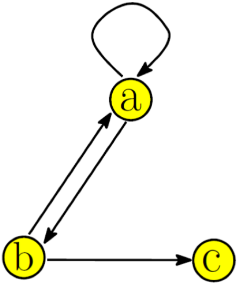
Is R reflexive?.



Reflexive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$

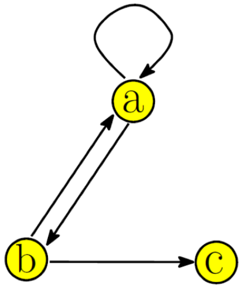
Is R reflexive?. *NO!*



Reflexive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a,a), (a,b), (b,a), (b,c)\} \subseteq A \times A$

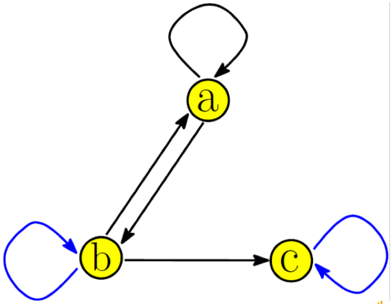
Is R reflexive?. NO!
What is the minimum set of pairs that we need to add to R to make it reflexive?



Reflexive Closure

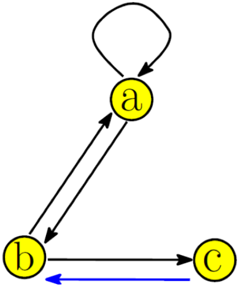
Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a,a), (a,b), (b,a), (b,c)\} \subseteq A \times A$

Is R reflexive?. NO!
What is the minimum set of pairs that we need to add to R to make it reflexive? $r(R) = R \cup \{(b,b), (c,c)\}$



Symmetric Closure

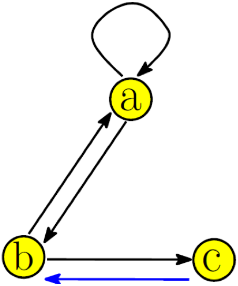
Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$



Symmetric Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$

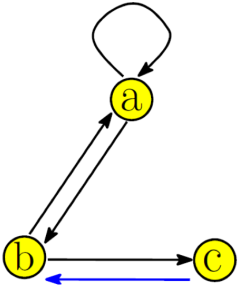
Is R symmetric?.



Symmetric Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$

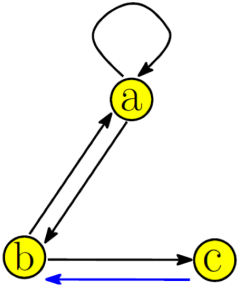
Is R symmetric?. NO!



Symmetric Closure

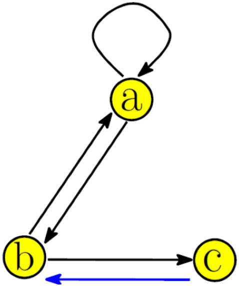
Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$

Is R symmetric?. NO!
What is the minimum set of pairs that we need to add to R to make it symmetric?



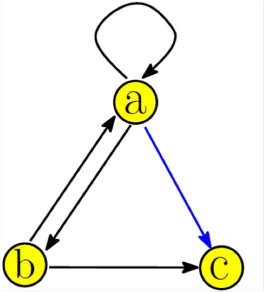
Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a,a), (a,b), (b,a), (b,c)\} \subseteq A \times A$

Is R symmetric?. NO!
What is the minimum set of pairs that we need to add to R to make it symmetric? $s(R) = R \cup \{(c,b)\}$



Transitive Closure

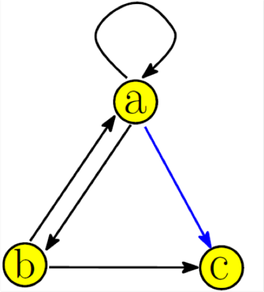
Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a,a), (a,b), (b,a), (b,c)\} \subseteq A \times A$



Transitive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a,a), (a,b), (b,a), (b,c)\} \subseteq A \times A$

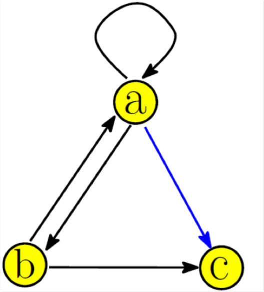
Is R Transitive?.



Transitive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$

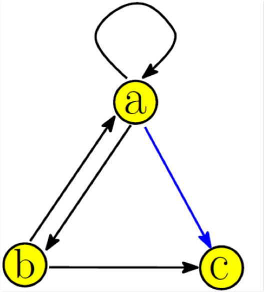
Is R Transitive?. NO!



Transitive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a,a), (a,b), (b,a), (b,c)\} \subseteq A \times A$

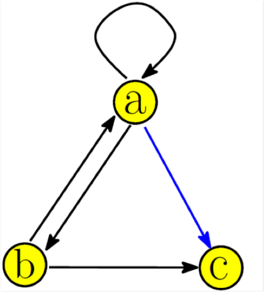
Is R Transitive?. NO!
What is the minimum set of pairs that we need to add to R to make it Transitive?



Transitive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a,a), (a,b), (b,a), (b,c)\} \subseteq A \times A$

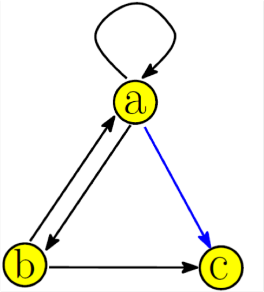
Is R Transitive?. NO!
What is the minimum set of pairs that we need to add to R to make it Transitive? $t(R) = R \cup \{(a,c)\}$



Transitive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$

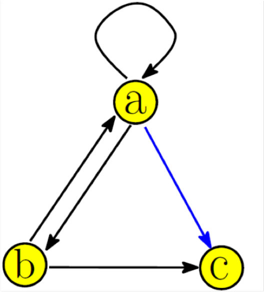
Is R Transitive?. NO!
What is the minimum set of pairs that we need to add to R to make it Transitive? $t(R) = R \cup \{(a, c)\}$
Is something missing?



Transitive Closure

Consider a set: $A = \{a, b, c\}$
Binary relation: $R = \{(a, a), (a, b), (b, a), (b, c)\} \subseteq A \times A$

Is R Transitive?. NO!
What is the minimum set of pairs that we need to add to R to make it Transitive? $t(R) = R \cup \{(a, c), (b, b)\}$
Is something missing? Since $(b, a), (a, b) \in R$, we should also have (b, b)



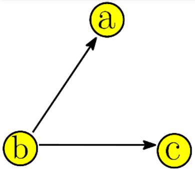
Closures - Practice Questions

Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.
Is it true or false?
The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

Yes. Can you show how?
For illustration, lets look at an example.

Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.
Is it true or false?
The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

Yes. Can you show how?
For illustration, lets look at an example.



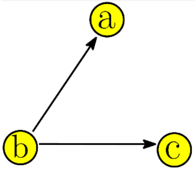
Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.

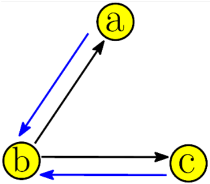
Is it true or false?

The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

Yes. Can you show how?

For illustration, lets look at an example.





Closures - Practice Questions

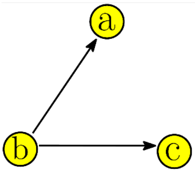
Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.

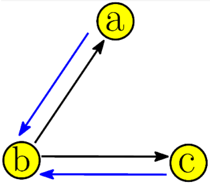
Is it true or false?

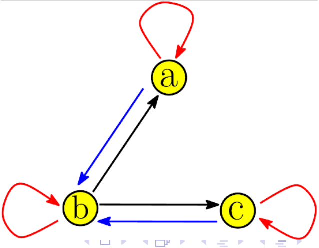
The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

Yes. Can you show how?

For illustration, lets look at an example.







Closures - Practice Questions

Let R be a relation such that $\forall a \in A, \exists b$ such that either aRb or bRa.
Is it true or false?
The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

1.

2. 3. 4.

Closures - Practice Questions

Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.

Is it true or false?

The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

1. We Want To Show (WTS): $\forall a \in A, (a, a) \in t(s(R))$.

2. 3. 4.

Closures - Practice Questions

Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.

Is it true or false?

The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

1. We Want To Show (WTS): $\forall a \in A, (a, a) \in t(s(R))$. OR $(c, c) \in t(s(R))$ for arbitrary c.

2. 3. 4.

Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.
Is it true or false?
The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

1. We Want To Show (WTS): $\forall a \in A, (a,a) \in t(s(R))$. OR $(c,c) \in t(s(R))$ for arbitrary c.
2. From Hypothesis: For an arbitrary $c \in A$, we have $(c,d) \in R$ for some d. if $c = d$, we are done.
3. 4.

Closures - Practice Questions

Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.
Is it true or false?
The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

1. We Want To Show (WTS): $\forall a \in A, (a, a) \in t(s(R))$. OR $(c, c) \in t(s(R))$ for arbitrary c.
2. From Hypothesis: For an arbitrary $c \in A$, we have $(c, d) \in R$ for some d. if $c = d$, we are done.
3. Case $c \neq d$: we have $(d, c) \in s(R)$ by definition of symmetric closure.
4.

Closures - Practice Questions

Let R be a relation such that $\forall a \in A$, $\exists b$ such that either aRb or bRa.
Is it true or false?
The transitive closure of the symmetric closure of R is also reflexive, that is, t(s(R)) is reflexive.

1. We Want To Show (WTS): $\forall a \in A, (a, a) \in t(s(R))$. OR $(c, c) \in t(s(R))$ for arbitrary c.
2. From Hypothesis: For an arbitrary $c \in A$, we have $(c, d) \in R$ for some d. if $c = d$, we are done.
3. Case $c \neq d$: we have $(d, c) \in s(R)$ by definition of symmetric closure.
4. As $(c, d) \in s(R), \wedge (d, c) \in s(R)$, we have $(c, c) \in t(s(R))$ by definition of transitive closure. Q.E.D.

What is Modular Arithmetic?

Modular arithmetic is a system of arithmetic for integers that considers only the remainders when dividing numbers.

Definition

For a positive integer m, two integers a and b are said to be congruent modulo m if they have the same remainder when divided by m. This is denoted as:

$$a \equiv b \pmod{m}$$

Addition in Modular Arithmetic

To perform addition in modular arithmetic, add the numbers as you normally would and then take the remainder when dividing by m.

Example

$$7 + 5 \equiv 12 \equiv 2 \pmod 5$$

Subtraction in Modular Arithmetic

To perform subtraction in modular arithmetic, subtract the numbers as you normally would and then take the remainder when dividing by m.

Example

$$11 - 8 \equiv 3 \pmod{5}$$

Multiplication in Modular Arithmetic

To perform multiplication in modular arithmetic, multiply the numbers as you normally would and then take the remainder when dividing by m.

Example

$$6 \cdot 4 \equiv 24 \equiv 4 \pmod{5}$$

Division in Modular Arithmetic

Division in modular arithmetic is a bit more complex. You need to find the modular multiplicative inverse.

Example

To find $12 \div 3 \pmod{7}$, you need to find the modular multiplicative inverse of 3 modulo 7, which is 5. Therefore:

$$12 \div 3 \equiv 12 \cdot 5 \equiv 60 \equiv 4 \pmod{7}$$

Equivalence Relation

We saw that relation on a set A is Reflexive, if it is Symmetric, and Transitive. Except for a trivial case!

Something funny happens when a relation on a set A is Reflexive, Symmetric, and Transitive at the same time. Can you guess what?

Equivalence Relation

We saw that relation on a set A is Reflexive, if it is Symmetric, and Transitive. Except for a trivial case!

Something funny happens when a relation on a set A is Reflexive, Symmetric, and Transitive at the same time. Can you guess what?

Parts (subsets) of set A sort of "clump" together into clusters.
The phenomenon is known Equivalence Relation.

Equivalence Relation

Formally, a relation on a set A is Equivalence, if it is Reflexive, Symmetric, and Transitive.

An Equivalence relation "partitions" the set A into parts called equivalence classes.

A Partition of a set A, is a family of subsets (each called a part) $A_1, A_2, \ldots, A_k$ such that

- They are non-empty. $\forall 1 \leq i \leq k, A_i \neq \emptyset$.
- They cover all A. $\bigcup_{1 \leq i \leq k}, A_i = A$.
- They don't overlap. $\forall 1 \leq i \neq j \leq k, A_i \cap A_j = \emptyset$.

Equivalence Relation: Examples

Consider the relation of equal cardinality on power set of a set A, i.e., For $X, Y \subseteq A$, aRb (or a b) whenever $|X| = |Y|$.

Is R an Equivalence?
How many equivalence classes?
What is largest one? And what is its size?

Consider the intersection relation on power set of a set A, i.e., For $X, Y \subseteq A$, aRb whenever $X \cap Y \neq \emptyset$.

- Is R an Equivalence?
- How many equivalence classes?
- What is largest one? And what is its size?

Equivalence Relation: Examples

Consider the relation of equal cardinality on power set of a set A, i.e., For $X, Y \subseteq A$, aRb (or a b) whenever $|X| = |Y|$.

Is R an Equivalence? YES!
How many equivalence classes?
What is largest one? And what is its size?

Consider the intersection relation on power set of a set A, i.e., For $X, Y \subseteq A$, aRb whenever $X \cap Y \neq \emptyset$.

- Is R an Equivalence?
- How many equivalence classes?
- What is largest one? And what is its size?

Equivalence Relation: Examples

Consider the relation of equal cardinality on power set of a set A, i.e., For $X, Y \subseteq A$, aRb (or a b) whenever $|X| = |Y|$.

Is R an Equivalence? YES!
How many equivalence classes? n+1
What is largest one? And what is its size?

Consider the intersection relation on power set of a set A, i.e., For $X, Y \subseteq A$, aRb whenever $X \cap Y \neq \emptyset$.

- Is R an Equivalence?
- How many equivalence classes?
- What is largest one? And what is its size?

Equivalence Relation: Examples

Consider the relation of equal cardinality on power set of a set A, i.e., For $X, Y \subseteq A$, aRb (or a b) whenever $|X| = |Y|$.

Is R an Equivalence? YES!
How many equivalence classes? n+1
What is largest one? And what is its size? $\binom{n}{\frac{n}{2}}$

Consider the intersection relation on power set of a set A, i.e., For $X, Y \subseteq A$, aRb whenever $X \cap Y \neq \emptyset$.

- Is R an Equivalence?
- How many equivalence classes?
- What is largest one? And what is its size?

Equivalence Relation: Examples

Consider the relation of equal cardinality on power set of a set A, i.e., For $X, Y \subseteq A$, aRb (or a b) whenever $|X| = |Y|$.

Is R an Equivalence? YES!
How many equivalence classes? n+1
What is largest one? And what is its size? $\binom{n}{\frac{n}{2}}$

Consider the intersection relation on power set of a set A, i.e., For $X, Y \subseteq A$, aRb whenever $X \cap Y \neq \emptyset$.

- Is R an Equivalence? NO!
- How many equivalence classes?
- What is largest one? And what is its size?

Equivalence Relation: Examples

Example: Let R be the relation on the set of real numbers such that $(a, b) \in R$ iff $a - b$ is an integer.

Question: Is R an equivalence relation?

Equivalence Relation: Examples

Example: Let R be the relation on the set of real numbers such that $(a, b) \in R$ iff $a - b$ is an integer.

Question: Is R an equivalence relation?

- As $a - a = 0$, which is an integer for every real numbers a. So, $(a, a) \in R$ for every real number a. Hence R is reflexive.
- Let $(a, b) \in R$, then $a - b$ is an integer, so $b - a$ is also an integer. Hence $(b, a) \in R$, i.e., R is symmetric.
- If (a, b) and $(b, c) \in R$, then $a - b$ and $b - c$ are integers. So, $a - c = (a - b) + (b - c)$ is also an integer. Hence, $(a, c) \in R$. Thus R is transitive

Consequently, R is an equivalence relation.

Equivalence Classes

If R is an equivalence relation over A, then for each $a \in A$ the equivalence class of a, denoted by $[a]$, is the set $[a] = x|xRa$.
An equivalence class has important properties:

- The equivalence classes over A form a partition of A.
- For every pair $a, b \in A$ we have either $[a] = [b]$, or $[a] \cap [b] = \emptyset$.

In other words, every element is in only one equivalence class.

Equivalence Classes

Define the relation, R on $\mathbb{Z}$, so that $\langle x, y \rangle \in R$ if and only if $x \mod 5 = y \mod 5$.

Is this an equivalence relation? Yes. (RST).

There are five equivalence classes under R corresponding to the five possible values $\mod 5$:

$$\{0, 5, 10, \ldots\},$$
$$\{1, 6, 11, \ldots\},$$
$$\{2, 7, 12, \ldots\},$$
$$\{3, 8, 13, \ldots\},$$
$$\{4, 9, 14, \ldots\}.$$

Inverse Relation

For a binary relation R from set A to B, an inverse relation R^{-1} is defined as a relation from set B to A where $bR^{-1}a$ iff aRb where $a \in A$ and $b \in B$.

- For example, if $R = \{(x,y),(y,z),(y,x)\}$ then $R^{-1} = \{(x,y),(z,y),(y,x)\}$.
- Practice: prove that if R is symmetric then R^{-1} is also symmetric.
- Practice: prove that $(R^{-1})^{-1} = R$

Point to ponder: If R is an equivalence relation, is it true that R^{-1} is also an equivalence relation?