Discrete Structures

Fight for Survival

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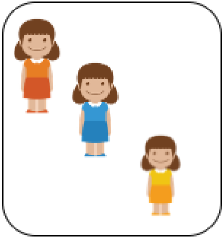
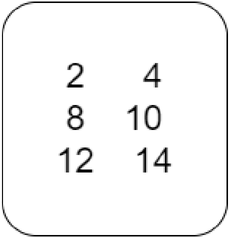
Introduction to sets

A set is an unordered collection of unique objects, called elements or members of the set.
A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

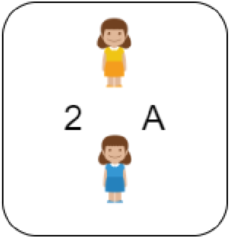
Set of girls in a class



Set of even numbers



A random set



Sets

Empty set: A set with no element is an empty set and denoted by ϕ .

Null set: The empty set is also referred to as the null set and can be denoted by $\{\}$.

Singleton: A set with a single element.

Finite set: A finite set has a finite number of elements.

Infinite set: An infinite set has an infinite number of elements.

Cardinality: The cardinality of a finite set A , denoted by $|A|$, is the number of elements in A.

Often Used Sets

These sets, each denoted using a boldface letter, play an important role in discrete mathematics:

$\mathbb{N} = \{0, 1, 2, 3, ...\}$, the set of natural numbers

$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$, the set of integers

$\mathbb{Z}^+ = \{1, 2, 3, ...\}$, the set of positive integers

$\mathbb{Q} = \{p/q \ |p \in \mathbb{Z}, q \in \mathbb{Z},$ and $q \neq 0\}$, the set of rational numbers

$\mathbb{R}:$, the set of real numbers

$\mathbb{R}^+:$ the set of positive real numbers

$\mathbb{C}:$ the set of complex number

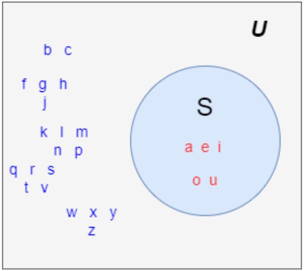
Universal Set

Universal Set U is the set of all elements from which given any set is drawn.

- For the set $\{-2, 0.4, 3\}$, U would be the set of real numbers, $\mathbb{R}$.
- For the set of vowels in alphabets, U would be the set of all the letters of alphabets.

Venn Diagrams

Sets can be represented graphically using Venn diagrams



- The box represents the Universal set.
- The circle represent sets(s).

Example

Write a Venn diagram representing sets of numbers: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

Subsets

We say that set A is a subset of B, denoted $A \subseteq B$, if every element of A is an element of B.

- If $S = \{2, 4, 6\}$, $T = \{1, 2, 3, 4, 5, 6, 7\}$,S is a subset of T. Denoted by $S \subseteq T$.
- For any set S, $S \subseteq S$
- For any set S, $\phi \subseteq S$

Example

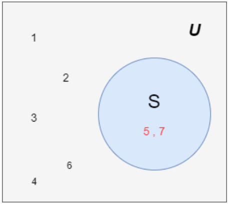
Let $A = \{x \in \mathbb{Z} | x = 6r + 12 \wedge r \in \mathbb{Z}\}$, and
$B = \{y \in \mathbb{Z} | y = 3r \wedge r \in \mathbb{Z}\}$
Prove $A \subseteq B$, and $B \nsubseteq A$.

Proper Subsets

If Set A is a subset of a set B but that $A \neq B$, we write "$A \subset B$" and say that A is a "proper subset" of B.

For $A \subset B$ to be true, it must be the case that $A \subseteq B$ and there must exist an element x of B that is not an element of A. That is, A is a proper subset of B if and only if

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$



Set of Sets or Family of Sets

Sets can contain other sets

$$S = \{\{1\}, \{2\}, \{3\}\}$$
$$T = \{\{\{1\}, \{\{2\}\}\}, \{\{3\}\}\}$$
$$V = \{\{\{1\}, \{\{2\}\}\}, \{\{\{3\}\}\}\}$$

V has only three elements

Equality of Sets

Two sets are equal if they have the same elements.

$\{1, 2, 3, 4, 5\} = \{5, 1, 3, 2, 4\}$
$\{1, 2, 3, 4, 5\} \neq \{1, 2, 3\}$

We can say set A is equal to set B $(i.e., A = B)$ if and only if $A \subseteq B$ and $B \subseteq A$

Example

Let $A = \{x \in \mathbb{Z} | x = 2r \wedge r \in \mathbb{Z}\}$, and $B = \{y \in \mathbb{Z} | y = 2s - 2 \wedge s \in \mathbb{Z}\}$
$A = B?$.

Set Cardinality

The cardinality of a set is the number of element in a set, written as $|A|$

$R = \{1, 2, 3, 4, 5, 6\}$, Then $|R| = 6$
$|\phi| = 0$
$S = \{\phi, 2, 3, 4, 6\}$ Then $|S| = 5$

What is the cardinality of the set $\{\phi\}$?

Power Set

The Power Set of S, written as $P(S)$ or 2^S is the set of all the subsets of S

$S = \{0, 1\}$,
All possible subsets of S :
ϕ (as it is the subset of all the sets), $\{0\}$, $\{1\}$, $\{0, 1\}$
$P(S) = \{\phi, \{0\}, \{1\}, \{0, 1\}\}$
Note : $|S| = 2$ and $|P(S)| = 4$

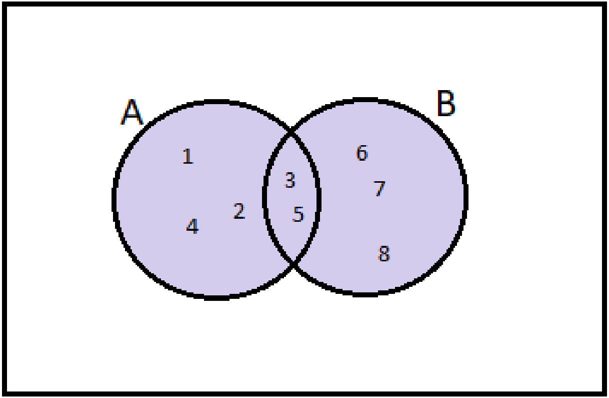
If a set has n elements, then the power set has 2^n elements

$$|P(S)| = 2^n$$

Set Operations : Union

The union of two sets, *A* and B, denoted $A \cup B$ is the set of all elements that are elements of *A* or B.

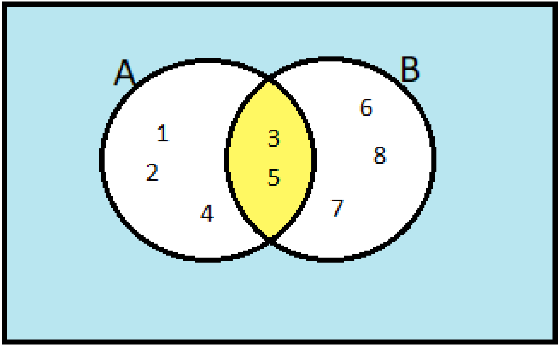
$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$



Set Operations : Intersection

The intersection of two sets A and B, denoted $A \cap B$ is the set of all elements that are elements of both A and B.

$$A \cap B = \{x|x \in A \text{ and } xB\}$$



Do it yourself!

Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

Do it yourself!

Let $A = \{0, 2, 4, 6, 8\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 3, 6, 9\}$. What are $A \cup B \cup C$ and $A \cap B \cap C$?

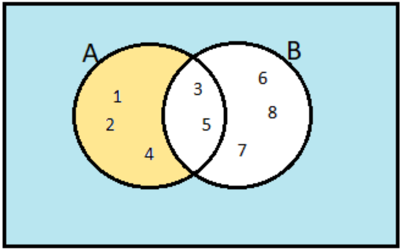
Solution:

The set $A \cup B \cup C$ contains those elements in at least one of A, B, and C. Hence, $A \cup B \cup C = \{0, 1, 2, 3, 4, 6, 8, 9\}$.
The set $A \cap B \cap C$ contains those elements in all three of A, B, and C. Thus, $A \cap B \cap C = \{0\}$.

Set Operations : Difference

The difference between two sets A and B, denoted as $A - B$, is the set of elements that are in A but not in B.

$$\{x | x \in A \text{ and } x \notin B\}$$



Set Operations : Difference

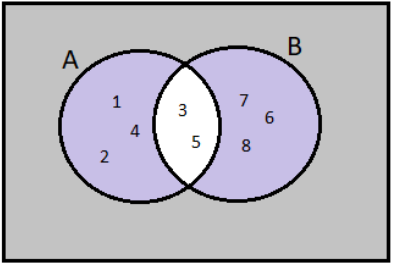
The difference of $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{5\}$; that is, $\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$.

This is different from the difference of $\{1, 2, 3\}$ and $\{1, 3, 5\}$, which is the set $\{2\}$.

Set Operations : Symmetric Difference

The symmetric difference between two sets, A and B, denoted $A \oplus B$, is the set of elements that are a member of exactly one of A and B, but not both

$\{x | x \in A$ or $x \in B$ but not both$\}$

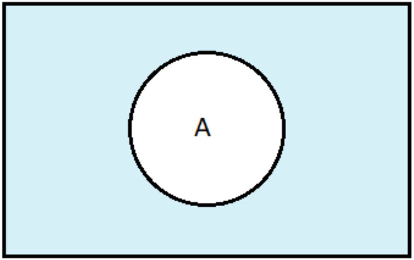


What is $A \oplus A$? What is $A \oplus A \oplus A$?

Set Operations: Complements

The Complement of a Set A is a set of all those elements of the universal set which do not belong to A and is denoted by A^c or A'

$$A' = U - A = \{x : x \in U \text{ and } x \notin A\} = \{x : x \notin A\}$$



Example

$A = (-1, 0] = \{x \in \mathbb{R} | -1 < x \leq 0\}, B = [0, 1) = \{x \in \mathbb{R} | 0 \leq x < 1\}$. Find $A \cup B, A \cap B, B - A$, and A'.

Set Operations: Notations

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \quad \text{Union of } n \text{ Sets}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n \quad \text{Intersection of } n \text{ Sets:}$$

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \ldots \quad \text{Union of Infinite Sets}$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \ldots \quad \text{Intersection of Infinite Sets:}$$

Set Operations: Notations

Example

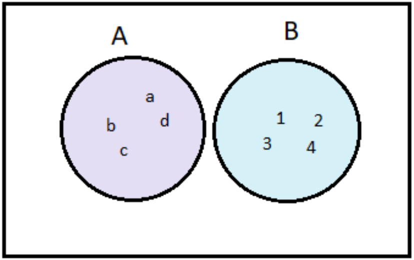
For each positive integer i, let

$$A_i = \{x \in \mathbb{R} | -\frac{1}{i} < x < \frac{1}{i}\} = \left(-\frac{1}{i}, \frac{1}{i}\right)$$

Find $A_1 \cup A_2 \cup A_3$, and $A_1 \cap A_2 \cap A_3$.
Find $\bigcup_{i=1}^{\infty} A_i$, and $\bigcap_{i=1}^{\infty} A_i$

Disjoint Sets

Two sets are disjoint if there intersection is empty set.



ϕ and ϕ are disjoint set, their intersection is empty set.

Set identities

Law Name	Identities	
Idempotent	$A \cup A = A$	$A \cap A = A$
Associative	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Commutative	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \phi = A$	$A \cap \cup = A$
Domination	$A \cap \phi = \phi$	$A \cup U = U$
Double Complement	$(A')' = A$	
Complement	$A \cap A' = \phi$, and $U' = \phi$	$A \cup A' = U$, and $\phi' = U$
DeMorgan's	$(A \cup B)' = A' \cap B'$	$(A \cap B)' = A' \cup B'$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$

Set identities-Example

$(A \cap B)' = \{x \mid x \notin A \cap B\}$	by definition of complement
$= \{x \mid -(x \in (A \cap B))\}$	by definition of does not belong
$= \{x \mid -(x \in A \wedge x \in B)\}$	by definition of intersection
$= \{x \mid -(x \in A) \vee -(x \in B)\}$	by the 1st De Morgan law for logical equivalences
$= \{x \mid x \notin A \vee x \notin B\}$	by definition of does not belong symbol
$= \{x \mid x \in A' \vee x \in B'\}$	by definition of complement
$= \{x \mid x \in A' \cup B'\}$	by definition of union
$= A' \cup B'$	by meaning of set builder notation

Tuples

In a 2-dimensional space,it is a (x,y) pair of numbers to specify a location.
In a 3-dimensional space,it is a (x,y,z) triple of numbers to specify a location.
A n-dimensional space, uses n-tuple of numbers.

- 2-D space uses pairs, or 2-tuple
- 3-D space uses triples, or 3-tuple

Note that these tuples are ordered, unlike sets.

Cartesian Product

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times$ B, is the set of all ordered pairs (a, b), where a $\in A$ and b $\in$B.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Note that the Cartesian products $A \times$ B and B $\times$ A are not equal, unless $A = \phi$ or $B = \phi$

Inclusion-Exclusion rules- Two Sets

Inclusion–Exclusion for sets:
The number of elements in the union of two sets is the sum of the number of elements in these sets minus the number of elements in their intersection.

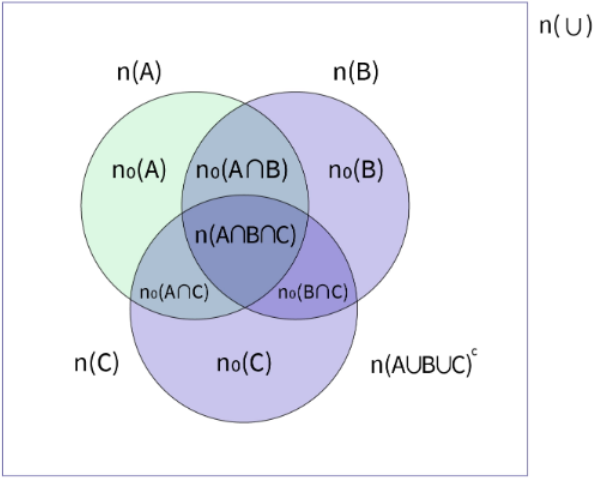
$$|S \cup T| = |S| + |T| - |S \cap T|$$

Inclusion-Exclusion rules- Three Sets

Consider three sets S_1, S_2, S_3 , for counting the total elements in the union of three sets , we will need to know the number of items in set S_1, the number of items in set S_2, the number of items in set S_3 and then we need to know the number of items common in all the three sets S_1, S_2 and S_3. (overlapping of all the three sets).

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_2 \cap S_3| - |S_3 \cap S_1| + |S_1 \cap S_2 \cap S_3|$$

Inclusion-Exclusion rules- Three Sets



Inclusion-Exclusion rules- Example

Consider a group of students out of which 15 study Potions, 11 study DADA and 13 study Charms. The number of students who study all three are 4. The number of students who study both Potions and DADA are 6, The number of students who study both Charms and Potions are 8 and The number of students who study both Charms and DADA are 7. What is the size of the group?

Considering the above scenario we need to find out how many students study either (Potions or DADA or Charms) or all.

Inclusion-Exclusion rules- Example

Lets us use P for Potions and C for DADA and E for Charms, we get the following formula:

$$|P \cup C \cup E| = |P| + |C| + |E| - |P \cap C| - |C \cap E| - |E \cap P| + |P \cap C \cap E|$$
$$|P \cup C \cup E| = 15 + 11 + 13 - 6 - 7 - 8 + 4$$
$$|P \cup C \cup E| = 22$$

Computer Representation of Sets

There are various ways to represent sets using a computer. One method is to store the elements of the set in an unordered fashion. However, if this is done, the operations of computing the union, intersection, or difference of two sets would be time-consuming, because each of these operations would require a large amount of searching for elements.

We will present a method for storing elements using an arbitrary ordering of the elements of the universal set. This method of representing sets makes computing combinations of sets easy.

Computer Representation of Sets

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order.

What bit strings represent the subset of all odd integers in U, the subset of all even integers in U, and the subset of integers not exceeding 5 in U?

Computer Representation of Sets

The set of odd integers in U, namely, {1, 3, 5, 7, 9}, has a one bit in the first, third, fifth, seventh, and ninth positions, and a zero elsewhere. It is 10 1010 1010.

Similarly, we represent the subset of all even integers in U, namely, {2, 4, 6, 8, 10}, by the string 01 0101 0101.

The set of all integers in U that do not exceed 5, namely,{1, 2, 3, 4, 5}, is represented by the string 11 1110 0000.